3 The Lorentz model for the classical transport of charges

3.1 Hypothesis of the model

Assumptions

1. $m_e \ll M_{\text{ion}}$
   - naming of distributions: f for e, F for ions.
2. ignore e^--e^- interactions
   - handwaving ... and later in the book
   - treating e^--density as small: $n_e^2 \ll n_e$
3. electron scattering is short range

   → it takes a very short time
   - e^- interacts with ions only "once", i.e. no double interactions

   ⇒ we can calculate the kinematics of the scattering by scattering theory (Appendix C)
4. there are no e^--ion correlations **before** the collision

3.2 Lorentz kinetic equation

The equation can be obtained from the BBGKY hierarchy

$$\left[\frac{\partial}{\partial t} + \{H_n, \cdot\}\right] F^{(1,0)}[\vec{r}_a, \vec{p}_a; t] = \int d\tilde{a} \frac{\partial \phi^{AA}_{a\tilde{a}}}{\partial \vec{r}_a} \frac{\partial}{\partial \vec{p}_a} F^{(2,0)}[\vec{r}_a, \vec{p}_a, \vec{r}_{\tilde{a}}, \vec{p}_{\tilde{a}}; t] + \int d\tilde{b} \frac{\partial \phi^{AB}_{a\tilde{b}}}{\partial \vec{r}_a} \frac{\partial}{\partial \vec{p}_a} F^{(1,1)}[\vec{r}_a, \vec{p}_a; \vec{r}_{\tilde{b}}, \vec{p}_{\tilde{b}}; t] \tag{1}$$

- from assumption 2. we get $F^{(2,0)} = 0$
- from assumption 4. we get $F^{(1,1)}[\vec{r}_a, \vec{p}_a; \vec{r}_{\tilde{b}}, \vec{p}_{\tilde{b}}; t] = f[\vec{r}_a, \vec{p}_a; t] F[\vec{r}_{\tilde{b}}, \vec{p}_{\tilde{b}}; t]$
- so going from $\vec{p} \to \vec{c}$ we get

$$\left[\frac{\partial}{\partial t} + \vec{c} \cdot \frac{\partial}{\partial \vec{r}} + \dot{\vec{c}} \cdot \frac{\partial}{\partial \vec{c}}\right] f[\vec{r}, \vec{c}; t] = \int d^3x_b\, d^3c_b\, F[\vec{x}_b, \vec{c}_b; t] \frac{\partial \phi^{AB}(\vec{r} - \vec{x}_b)}{\partial \vec{r}} \cdot \frac{\partial f[\vec{r}, \vec{c}; t]}{\partial \vec{c}} \tag{2}$$

- from assumption 3. (scattering theory) we can determine the kinematics of the scattering **instead** of using the term $\frac{\partial \phi^{AB}(\vec{r}-\vec{x}_b)}{\partial \vec{r}} \cdot \frac{\partial}{\partial \vec{c}} \quad \propto \delta(\vec{r} - \vec{x}_b)$
  - scattering happens only at a single point: $f[\vec{r}_a, \vec{c}_a; t] F[\vec{r}_{\tilde{b}}, \vec{c}_{\tilde{b}}; t] \to f[\vec{r}, \vec{c}_a; t] F[\vec{r}, \vec{c}_{\tilde{b}}; t]$
- the integral $\int d^3x_b\, d^3c_b\, F[\vec{x}_b, \vec{c}_b; t]$ counts the scattering centers (ions)

⇒ the r.h.s. can be understood as terms that reduce the number of electrons in the range $(c, c + \delta c)$, i.e. **loss terms** $J_-[f]$, and terms that increase this number, i.e. **gain terms** $J_+[f]$.

Scattering: $\vec{c}_{[e^-]} + \vec{c}_{1\,[\text{ion}]} \to \vec{c}\,'_{[e^-]} + \vec{c}\,'_{1\,[\text{ion}]}$ loss term

$$d^3c\, J_-[f(\vec{r}, \vec{c}; t)] = \int f(\vec{r}, \vec{c}; t) |\vec{c} - \vec{c}_1| \Delta t\, b\, db\, d\psi\, d^3c\, F(\vec{r}, \vec{c}_1; t) d^3r\, d^3c_1 \tag{3}$$

Inverse scattering: $\vec{c}^*_{[e-]} + \vec{c}^*_{1\,[\text{ion}]} \to \vec{c}_{[e-]} + \vec{c}_{1\,[\text{ion}]}$ gain term

$$d^3c^*\, J_+[f(\vec{r},\vec{c}^*;t)] = \int f(\vec{r},\vec{c}^*;t)|\vec{c}^* - \vec{c}_1^*|\Delta t\, b^*\, db^*\, d\psi^*\, d^3c^*\, F(\vec{r},\vec{c}_1^*;t) d^3r\, d^3c_1^* \tag{4}$$

- time reversal invariance of classical scattering exchanges initial and final states between J_+ and J_-:
  - $\vec{c}^* \leftrightarrow \vec{c}'$ and $\vec{c}_1^* \leftrightarrow \vec{c}_1'$
  - energy, momentum, and angular momentum conservation imply the same impact parameter and azimutal angle: $b^* = b$ and $\psi^* = \psi$
  - the integral measure for the initial state is the same as for the final state: $d^3c\, d^3c_1 = d^3c'\, d^3c_1'$
    - can be seen by going to Jacobi coordinates
- so get the gain term as

$$d^3c\, J_+[f(\vec{r},\vec{c}';t)] = \int f(\vec{r},\vec{c}';t)|\vec{c} - \vec{c}_1|\Delta t\, b\, db\, d\psi\, d^3c\, F(\vec{r},\vec{c}_1';t) d^3r\, d^3c_1 \tag{5}$$

- and with interpreting $\dot{\vec{c}} = \frac{q}{m}\vec{E}$ we get (3.1) as the first equation of the BBGKY hierarchy:

$$\left[\frac{\partial}{\partial t} + \vec{c}\cdot\frac{\partial}{\partial \vec{r}} + \frac{q}{m}\vec{E}\cdot\frac{\partial}{\partial \vec{c}}\right] f[\vec{r},\vec{c};t] = \int [f'F' - fF]|\vec{c} - \vec{c}_1|\, b\, db\, d\psi\, d^3c_1 \tag{3.1}$$

  where $f = f(\vec{c}) = f(\vec{r},\vec{c};t)$, $f' = f(\vec{c}')$, $F = F(\vec{c}_1) = f(\vec{r},\vec{c}_1;t)$, and $F' = F(\vec{c}_1')$.

3.3 Ion distribution function

Ions in thermal equilibrium have the Maxwell-Boltzmann distribution

$$F[\vec{r},\vec{c}_1;t] = F_{\text{MB}}[\vec{c}_1] \;=\; n_i(\vec{r};t)\left(\frac{M}{2\pi k_B T}\right)^{3/2} \exp\left[-\frac{M\vec{c}_1^{\,2}}{2k_B T}\right] \tag{6}$$

with the ion density n_i and their mass M.

- for $M \gg m_e$ the distribution becomes much narrower:

$$\lim_{M\to\infty} F[\vec{r},\vec{c}_1;t] \;\to\; n_i(\vec{r};t)\delta(\vec{c}_1) \tag{7}$$

3.4 Equilibrium distribution

Energy conservation

$$\tfrac{1}{2}mc^2 + \tfrac{1}{2}Mc_1^2 = \tfrac{1}{2}mc'^{\,2} + \tfrac{1}{2}Mc_1'^{\,2} \tag{8}$$

implies

$$f_{\text{MB}}[\vec{c}']F_{\text{MB}}[\vec{c}_1'] = f_{\text{MB}}[\vec{c}]F_{\text{MB}}[\vec{c}_1] \tag{9}$$

and the r.h.s. of eq. (3.1) vanishes.

3.5 Conservation laws and collisional invariants

How microscopic conservation laws are expressed at the kinetic level.

Take any function $\varphi(\vec{c})$, multiply eq.(3.1) with it, and integrate over d^3c:

$$\int d^3c\varphi(\vec{c})\left[\frac{\partial}{\partial t} + \vec{c}\cdot\frac{\partial}{\partial \vec{r}} + \frac{q}{m}\vec{E}\cdot\frac{\partial}{\partial \vec{c}}\right] f[\vec{r},\vec{c};t] = \int [f'F' - fF]|\vec{c} - \vec{c}_1|\varphi(\vec{c})\, b\, db\, d\psi\, d^3c\, d^3c_1 \tag{10}$$

Using the definitions of Chapter 1 for density ρ_φ and flux $\vec{J}_\varphi$

$$\rho_\varphi = \int d^3c\, \varphi f \qquad \vec{J}_\varphi = \int d^3c\, \varphi f \vec{c} \tag{3.4}$$

- the first term of the l.h.s. is
$$\int d^3c\varphi\frac{\partial f}{\partial t} = \frac{\partial}{\partial t}\int d^3c\varphi f - \int d^3cf\frac{\partial\varphi}{\partial t} =: \frac{\partial}{\partial t}\rho_\varphi \tag{11}$$
since $\frac{\partial\varphi}{\partial t} = 0$.
- the second term of the l.h.s. is
$$\int d^3c\varphi\vec{c}\cdot\vec{\nabla}f = \vec{\nabla}\int d^3c\varphi\vec{c}f - \int d^3cf\vec{\nabla}(\varphi\vec{c}) =: \vec{\nabla}\vec{J}_\varphi \tag{12}$$
since $\vec{\nabla}(\varphi(\vec{c})\vec{c}) = 0$.
- the third term of the l.h.s. is
$$\begin{aligned}\int d^3c\varphi\frac{q}{m}\vec{E}\cdot\frac{\partial}{\partial\vec{c}}f = & \int d^3c\frac{\partial}{\partial\vec{c}}(f\varphi\frac{q}{m}\vec{E}) - \frac{q}{m}\vec{E}\cdot\int d^3cf\frac{\partial\varphi}{\partial\vec{c}}\\ =: & \left[f\varphi(\vec{c})\frac{q}{m}\vec{E}\right]_{\text{boundary}} - \frac{q}{m}\vec{E}\cdot\int d^3cf\vec{g} \;=\; 0 - \frac{q}{m}\vec{E}\cdot\vec{\rho}_{\vec{g}}\end{aligned} \tag{13}$$
where $\vec{g} = \frac{\partial\varphi}{\partial\vec{c}}$.

we get eq. (3.3):
$$\frac{\partial\rho_\varphi}{\partial t} + \vec{\nabla}\vec{J}_\varphi - \frac{q}{m}\vec{E}\cdot\vec{\rho}_{\vec{g}} = \int[f'F' - fF]|\vec{c}-\vec{c}_1|\,\varphi\, b\, db\, d\psi\, d^3c\, d^3c_1 \tag{3.3}$$
Looking at the r.h.s. we can use the same arguments that lead from eq. (4) to eq. (5) to reformulate the part with the primed distributions into unprimed ones, giving
$$\frac{\partial\rho_\varphi}{\partial t} + \vec{\nabla}\vec{J}_\varphi - \frac{q}{m}\vec{E}\cdot\vec{\rho}_{\vec{g}} = \int[\varphi(\vec{c}') - \varphi(\vec{c})]\, fF\,|\vec{c}-\vec{c}_1|\, b\, db\, d\psi\, d^3c\, d^3c_1 \tag{3.7}$$

⇒ only differences of the function between initial and final state of the collision enter as sources for the evolution of the distributions!

!!! if the function describes an invariant of the collision ⇒ the r.h.s is always zero!

- taking as an example the charge of the particle: $\phi(\vec{c}) = q$ we have $\vec{g} = \frac{\partial q}{\partial\vec{c}} = 0$ and we get the differential form of Kirchhoff's law:
$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\vec{J} = 0 \tag{3.8}$$
where we have the charge density
$$\rho(\vec{r},t) = q\int d^3c\, f(\vec{r},\vec{c},t) \tag{3.9}$$
and the electric current
$$\vec{J}(\vec{r},t) = q\int d^3c\, f(\vec{r},\vec{c},t)\vec{c} \tag{3.10}$$

3.6 Kinetic collision models

3.6.1 Rigid hard sphere

Assumption: $M_{\text{ion}} \to \infty$, therefore static and $F(\vec{c}_1) = n_i\delta(\vec{c}_1)$.

The scattering can then be visualized by Fig. 3.5, giving the velocity after the collision as
$$\vec{c}' = \vec{c} - 2(\vec{c}\cdot\hat{n})\hat{n}\;\; . \tag{3.11}$$
The impact parameter is then just
$$b = R\sin\theta\;\; , \tag{14}$$

but the angle θ is restricted to $(-\frac{\pi}{2}, \frac{\pi}{2}]$. This gives the loss term, eq. (3),

$$J_-[f(\vec{r},\vec{c};t)] = \int f(\vec{r},\vec{c};t)|\vec{c}-\vec{c}_1|b\,db\,d\psi\,F(\vec{r},\vec{c}_1;t)d^3r\,d^3c_1 = n_i f(\vec{c})|\vec{c}|\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} R\sin\theta\,R\cos\theta\,d\theta\,2\pi$$
$$= n_i\pi R^2 f(\vec{c})|\vec{c}| \tag{15}$$

The rigid hard sphere model **assumes** an isotropic distribution of postcollisional velocities:

$$J_+[f(\vec{r},\vec{c};t)] = n_i\pi R^2|\vec{c}|\,\mathbb{P}f(\vec{c}) \ , \tag{16}$$

where $\mathbb{P}$ averages over the direction:

$$\mathbb{P}f(\vec{c}) = \frac{1}{4\pi}\int f(\vec{c})d^2\hat{c} \ . \tag{17}$$

3.6.2 Thermalizing ions: the BGK model

Allowing for a thermal distribution of ions, while still assuming similar kinematics: $|\vec{c}_1| \ll |\vec{c}|$ or $|\vec{c}-\vec{c}_1| \approx |\vec{c}|$. Treating ions as hard spheres gives the same loss term, only the gain term should be modified to reflect the impact of the ion distribution: $J_+ \propto |\vec{c}|\widehat{f}_{\rm MB}(\vec{c})$. Considering charge conservation gives the normalization

$$J_+ - J_- = n_i\pi R^2|\vec{c}|\left[\widehat{f}_{\rm MB}(\vec{c})\frac{\int|\vec{c}'|f(\vec{c}')d^3c'}{\int|\vec{c}'|\widehat{f}_{\rm MB}(\vec{c}')d^3c'} - f(\vec{c})\right] \ . \tag{3.17}$$

3.7 Electrical conduction

3.7.1 Conservation equation

We already got the continuity equation, eq. (3.8), but we now also look for the response to the electric field $\vec{E}$. The task is to derive Ohm's law $\vec{J} = \sigma\vec{E}$.

3.7.2 Linear response

Assumptions

- $\vec{E}$ is small: $\vec{E} = \epsilon\vec{E}_0$
- for $\epsilon \to 0$, the electrons will reach the equilibrium distributiuon $f_{\rm MB}$
- for small fields we can approximate (i.e. make the ansatz)

$$f(\vec{c}) = f_{\rm MB}(\vec{c})[1+\epsilon\Phi(\vec{c})] \tag{3.18}$$

- since Φ should be linear in $\vec{E}$, it has to the proportional to $\vec{E}$.
- symmetry (i.e. tensorial) analysis:

$$\Phi(\vec{c}) = \phi(c)\,\vec{c}\cdot\vec{E}_0 \tag{3.19}$$

  - as $\vec{c}$ is the only available quantity, that can combine with $\vec{E}_0$ to give a scalar.
  - and $\phi(c)$ depends only on $|\vec{c}|$.

3.7.3 Ohm's law

Using the ansatz eq. (3.18) with eq. (3.19) for the calculation of the current, eq. (3.10), we get

$$\vec{J}_j(\vec{r},t) = q\int d^3c\,\vec{c}_j f_{\rm MB}(\vec{c})[1+\epsilon\phi(c)\,\vec{c}\cdot\vec{E}] = q\int d^3c\,\vec{c}_j\,\vec{c}_k f_{\rm MB}(\vec{c})\phi(c)\epsilon\vec{E}_k = q\vec{\vec{\sigma}}_{jk}\cdot\epsilon\vec{E}_k \tag{3.20}$$

with the conductivity tensor $\vec{\vec{\sigma}}_{jk}$. For an isotropic system, the tensor is proportional to the unit matrix δ_{jk}, giving the scalar conductivity

$$\sigma = \tfrac{1}{3}{\rm Tr}\vec{\vec{\sigma}} = \tfrac{q}{3}\int d^3c\,|\vec{c}|^2 f_{\rm MB}(\vec{c})\phi(c) \ . \tag{3.21}$$

3.7.4 Electrical conductivity

...to actually calculate it, we need $\phi(c)$.

Inserting the ansatz, eq. (3.18) and eq. (3.19), into eq. (3.1), we get

$$\left[\frac{\partial}{\partial t} + \vec{c}\cdot\frac{\partial}{\partial \vec{r}} + \frac{q}{m}\vec{E}\cdot\frac{\partial}{\partial \vec{c}}\right] f_{\rm MB}(\vec{c})[1+\epsilon\phi(c)\,\vec{c}\cdot\vec{E}_0]$$
$$= \int \Big[f_{\rm MB}(\vec{c}')[1+\epsilon\phi(c')\,\vec{c}'\cdot\vec{E}_0]F' - f_{\rm MB}(\vec{c})[1+\epsilon\phi(c)\,\vec{c}\cdot\vec{E}_0]F\Big]\,|\vec{c}-\vec{c}_1|\,b\,db\,d\psi\,d^3c_1 \qquad (18)$$

The first two terms on the l.h.s. vanish, as our ansatz has no time dependence and no spatial dependence. Remembering $\vec{E}=\epsilon\vec{E}_0$, we see that the third term is already of order ϵ^1 (for $f_{\rm MB}$ see eq. (6):

$$\frac{q}{m}\vec{E}\cdot\frac{\partial}{\partial\vec{c}}f_{\rm MB}(\vec{c})[1+\epsilon\phi(c)\,\vec{c}\cdot\vec{E}_0] = \epsilon\frac{q}{m}\vec{E}_0\cdot\frac{\partial f_{\rm MB}(\vec{c})}{\partial\vec{c}} + \mathcal{O}(\epsilon^2) = \epsilon\frac{q}{m}\vec{E}_0\cdot\left(-\frac{m\vec{c}}{k_BT}\right)f_{\rm MB}(\vec{c}) + \mathcal{O}(\epsilon^2)\ . \qquad (19)$$

The r.h.s. is (again writing the shorter function Φ)

$$\epsilon^0\int\left[f_{\rm MB}(\vec{c}')F' - f_{\rm MB}(\vec{c})F\right]|\vec{c}-\vec{c}_1|\,b\,db\,d\psi\,d^3c_1$$
$$+\epsilon^1\int\left[f_{\rm MB}(\vec{c}')\Phi(\vec{c}')F' - f_{\rm MB}(\vec{c})\Phi(\vec{c})F\right]|\vec{c}-\vec{c}_1|\,b\,db\,d\psi\,d^3c_1 \qquad (20)$$

For the equilibrium distributions, depending only on $|\vec{c}|$, the ϵ^0-term vanishes, as $F'=F$ and $f'_{\rm MB}=f_{\rm MB}$, giving the simpler form:

$$\epsilon^1\int f_{\rm MB}F_{\rm MB}\left[\Phi(\vec{c}')-\Phi(\vec{c})\right]|\vec{c}-\vec{c}_1|\,b\,db\,d\psi\,d^3c_1\ . \qquad (3.23)$$

That inspires to **define** the linear integral operator (with switched ordering of primed/unprimed)

$$I[\Psi] := \int \widehat{f}_{\rm MB}\widehat{F}_{\rm MB}\left[\Psi(\vec{c})-\Psi(\vec{c}')\right]|\vec{c}-\vec{c}_1|\,b\,db\,d\psi\,d^3c_1\ , \qquad (3.24)$$

where the hatted functions are the velocity distributions normalized to 1 without the spatial dependence: $f_{\rm MB}=n_e\widehat{f}_{\rm MB}$. Then we can rewrite eq. (18) to order ϵ^1 as

$$\frac{q}{k_BT}(-\vec{c}\cdot\vec{E}_0)n_e\widehat{f}_{\rm MB}(\vec{c}) = -n_en_iI[\Phi] = -n_en_iI[\phi(c)\,\vec{c}\cdot\vec{E}_0]\ . \qquad (21)$$

Using isotropy we can drop $\vec{E}_0$, and defining a reduced $\hat{\phi}:=\frac{n_i}{q}\phi$, we get

$$\frac{n_i}{q}I[\phi(c)\vec{c}] = I[\hat{\phi}(c)\vec{c}] = \frac{1}{k_BT}\widehat{f}_{\rm MB}(c)\vec{c}\ , \qquad (3.26)$$

giving the conductivity, eq. (3.21):

$$\sigma = \frac{q}{3}\int d^3c\,|\vec{c}|^2\,f_{\rm MB}(\vec{c})\phi(c) = \frac{q}{3}\int d^3c\,c^2\,n_e\widehat{f}_{\rm MB}(\vec{c})\frac{q}{n_i}\hat{\phi}(c) = \frac{q^2n_e}{3n_i}\int d^3c\,c^2\,\widehat{f}_{\rm MB}(c)\hat{\phi}(c)\ . \qquad (3.27)$$

Rigid hard sphere

The integral operator, eq. (3.24), is basically $J_- - J_+$, which we calculated for the rigid hard sphere model: eq. (15) and eq. (16), allowing us to determine the $\hat{\phi}$ by eq. (3.26). For the gain term, we have to average over the postcollisional velocities. But integrating $\Psi=\phi(c)\vec{c}$ over the directions gives zero, so only J_- contributes and we get the equation

$$J_- - J_+ = n_i\pi R^2 f(\vec{c})|\vec{c}| = n_in_e\pi R^2\widehat{f}_{\rm MB}(c)\frac{q}{n_i}\hat{\phi}(c)\vec{c}\,|\vec{c}| \qquad (15)$$
$$= n_in_eI[\Psi] = n_in_eI[\phi(c)\vec{c}] = qn_eI[\hat{\phi}(c)\vec{c}] = \frac{qn_e}{k_BT}\widehat{f}_{\rm MB}(c)\vec{c}\ , \qquad (3.26)$$

giving the solution

$$\hat{\phi}(c) = \frac{1}{\pi R^2k_BT|\vec{c}|}\ . \qquad (22)$$

Putting this expression into the equation for the conductivity, eq. (3.27),

$$\begin{aligned}\sigma &= \frac{q^2 n_e}{3n_i}\int d^3c\, c^2 \left(\frac{m}{2\pi k_BT}\right)^{3/2} \exp\left[-\frac{m\vec{c}^2}{2k_BT}\right]\frac{1}{\pi R^2 k_BT|\vec{c}|} \\ &= \frac{q^2 n_e}{3n_i\pi R^2 k_BT}\left(\frac{m}{2\pi k_BT}\right)^{3/2}\int_0^\infty 4\pi dc\, c^3 \exp\left[-\frac{mc^2}{2k_BT}\right] \\ &= \frac{q^2 n_e}{3n_i\pi R^2 k_BT}\left(\frac{m}{2\pi k_BT}\right)^{3/2}\int_0^\infty 2\pi\frac{2k_BT}{m}dx\,\frac{2k_BT}{m}x\,e^{-x} \\ &= \frac{q^2 n_e}{3n_i k_BT\pi R^2}\sqrt{\frac{8k_BT}{\pi m}}\cdot 1 = \frac{q^2 n_e \ell}{3k_BT}\langle|\vec{c}|\rangle \ , \end{aligned} \tag{3.29}$$

with the mean free path $\ell = [n_i\pi R^2]^{-1}$ and the average velocity $\langle|\vec{c}|\rangle = \sqrt{\frac{8k_BT}{\pi m}}$.

BGK model

Due to isotropy, the BGK model gives the same solution as the rigid hard sphere model. For other model, one has to look for numeric solutions.

3.7.5 Frequency response

Taking the electric field as

$$\vec{E}(t) = \epsilon\vec{E}_0 e^{-i\omega t} \ , \tag{23}$$

the linear response will have the same forced time dependence

$$f(\vec{c},t) = f_{\rm MB}(c)[1+\epsilon\Phi_\omega(\vec{c})e^{-i\omega t}] \ , \tag{24}$$

giving an additional term, when inserted into eq. (3.1):

$$\begin{aligned}&\left[\frac{\partial}{\partial t} + \vec{c}\cdot\frac{\partial}{\partial\vec{r}} + \frac{q}{m}\epsilon\vec{E}_0 e^{-i\omega t}\cdot\frac{\partial}{\partial\vec{c}}\right] f_{\rm MB}(c)[1+\epsilon\Phi_\omega(\vec{c})e^{-i\omega t}] \\ &\quad = \epsilon\int f_{\rm MB}(c)F_{\rm MB}[\Phi_\omega(\vec{c}\,') - \Phi_\omega(\vec{c})]e^{-i\omega t}|\vec{c}-\vec{c}_1|\, b\, db\, d\psi\, d^3c_1 = -\epsilon n_i n_e I[\Phi_\omega]e^{-i\omega t} \ . \end{aligned} \tag{25}$$

The term ϵ^0 vanishes, and ϵ^1 gives

$$-i\omega f_{\rm MB}(c)\Phi_\omega(\vec{c})e^{-i\omega t} + \frac{q}{m}\epsilon\vec{E}_0 e^{-i\omega t}\cdot(-\frac{m\vec{c}}{k_BT})f_{\rm MB}(c) = -n_i n_e I[\Phi_\omega]e^{-i\omega t} \ . \tag{26}$$

Using linearity and isotropy we can make the ansatz

$$\Phi_\omega(\vec{c}) = \frac{q}{n_i}\widehat{\phi}_\omega\vec{c}\cdot\vec{E}_\omega \ , \tag{27}$$

resulting in

$$\frac{1}{k_BT}\widehat{f}_{\rm MB}(c)\vec{c} = I[\widehat{\phi}_\omega\vec{c}] - \frac{i\omega}{n_i}\widehat{f}_{\rm MB}(c)\widehat{\phi}_\omega\vec{c} \ , \tag{3.32}$$

which has a complex solution for the perturbation $\widehat{\phi}_\omega$, which gives a complex electrical conductivity, when integrating the get the current:

$$\begin{aligned}\vec{J}_j &= q\int d^3c\,\vec{c}_j f_{\rm MB}(c)[1+\epsilon\frac{q}{n_i}\widehat{\phi}_\omega\vec{c}\cdot\vec{E}_\omega e^{-i\omega t}] = \left[\frac{\epsilon q^2}{n_i}\int d^3c\,\vec{c}_j\vec{c}_k f_{\rm MB}(c)\widehat{\phi}_\omega\right]\cdot\vec{E}_{\omega\,k}e^{-i\omega t} \\ &= [\sigma_0 + i\sigma_1]_{jk}\vec{E}_{\omega\,k}e^{-i\omega t} = (\vec{\vec{\sigma}}_\omega\cdot\vec{E}_\omega e^{-i(\omega t-\alpha)})_j \ , \end{aligned} \tag{3.33}$$

→ exercise 3.10.

3.8 Relaxation dynamics

Assumptions

- near equilibrium
- no $\vec{E}$-field
- spatial dependence described with Fourier analysis:

$$f(\vec{r},\vec{c},t) = f_{\mathrm{MB}}(\vec{c})[1+\Phi_{\vec{k}}(\vec{c},t)e^{i\vec{k}\cdot\vec{r}}] \tag{3.34}$$

- and $\Phi_{\vec{k}}(\vec{c},t) \ll 1$

Insering into eq. (3.1) gives the l.h.s.

$$\left[\frac{\partial}{\partial t} + \vec{c}\cdot\frac{\partial}{\partial \vec{r}}\right] f_{\mathrm{MB}}(\vec{c})[1+\Phi_{\vec{k}}(\vec{c},t)e^{i\vec{k}\cdot\vec{r}}] = f_{\mathrm{MB}}(\vec{c})\left[\frac{\partial \Phi_{\vec{k}}(\vec{c},t)}{\partial t} + i\vec{k}\cdot\vec{c}\,\Phi_{\vec{k}}(\vec{c},t)\right] e^{i\vec{k}\cdot\vec{r}} \tag{28}$$

and the r.h.s.

$$\int f_{\mathrm{MB}}(\vec{c})F_{\mathrm{MB}}(\vec{c}_1)[\Phi_{\vec{k}}(\vec{c}',t)-\Phi_{\vec{k}}(\vec{c},t)]e^{i\vec{k}\cdot\vec{r}}|\vec{c}-\vec{c}_1|\,b\,db\,d\psi\,d^3c_1 = -\,n_i n_e I[\Phi_{\vec{k}}]e^{i\vec{k}\cdot\vec{r}} \tag{29}$$

Dropping the exponential and moving the second term of eq. (28) to eq. (29), we can define the linear operator

$$L_{\vec{k}}\Phi_{\vec{k}} := n_e n_i I[\Phi_{\vec{k}}] + i\vec{k}\cdot\vec{c}\,\Phi_{\vec{k}}(\vec{c},t) \ , \tag{30}$$

which gives the linear differential equation for the modes

$$f_{\mathrm{MB}}(\vec{c})\frac{\partial \Phi_{\vec{k}}(\vec{c},t)}{\partial t} = -L_{\vec{k}}\Phi_{\vec{k}} \ . \tag{3.35}$$

Making an ansatz for the seperation of variables with an exponentially decaying solution

$$\Phi_{\vec{k}}(\vec{c},t) = \Phi_{\vec{k}}(\vec{c})e^{\lambda t} \ , \tag{31}$$

one gets as a result the generalised eigenvalue equation

$$L_{\vec{k}}\Phi_{\vec{k}} = \lambda f_{\mathrm{MB}}(\vec{c})\Phi_{\vec{k}} \ . \tag{3.36}$$

3.8.1 Properties of the linear operator

Defining the scalar product of functions in the normal way:

$$(g,h) := \int g^*(\vec{c})h(\vec{c})d^3c \ . \tag{3.37}$$

Hermiticity With the same arguments as in sec. 3.5, i.e. the comments between eq. (4) and eq. (5), we can show:

$$(g,L_0h) = \int f_{\mathrm{MB}}(\vec{c})F_{\mathrm{MB}}(\vec{c}_1)g^*(\vec{c})[h(\vec{c}')-h(\vec{c})]|\vec{c}-\vec{c}_1|\,b\,db\,d\psi\,d^3c_1\,d^3c \tag{3.38}$$

$$= \int f_{\mathrm{MB}}(\vec{c})F_{\mathrm{MB}}(\vec{c}_1)[g^*(\vec{c}')-g^*(\vec{c})]h(\vec{c})|\vec{c}-\vec{c}_1|\,b\,db\,d\psi\,d^3c_1\,d^3c = (L_0g,h) \ , \tag{3.39}$$

proving that L_0 is a hermitian operator.

Positivity But applying the above arguments for only 1/2 of the scalar product, we can show

$$(g,L_0g) = \tfrac{1}{2}\int f_{\mathrm{MB}}(\vec{c})F_{\mathrm{MB}}(\vec{c}_1)[g^*(\vec{c}')-g^*(\vec{c})][g(\vec{c}')-g(\vec{c})]|\vec{c}-\vec{c}_1|\,b\,db\,d\psi\,d^3c_1\,d^3c \geq 0 \ , \tag{3.40}$$

which is already obvious from the hermiticity.

3.8.2 Kinetic gap

Since we have a collisional invariant, like the charge, where $g = \text{const} \equiv 1$, we do encounter the eigenvalue $\lambda_0 = 0$. For $\vec{k} = 0$ we get them for arbitrary scattering, but the other eigenvalues depend on the model.

On general grounds, one can prove, that the eigenvalues for $\vec{k} \to 0$ are **not** dense at 0, meaning:

$$\forall \vec{k} > 0 \quad \exists \Delta_{(\vec{k})} > 0 \ , \quad \text{so that} \quad \lambda_1(\vec{k}) \geq \Delta_{(\vec{k})} |\vec{k}| \ . \tag{32}$$

Consequence: any distribution that is not Maxwellian will decay to a Maxwellian distribution, as $\Phi = \text{const}$ gives exactly the Maxwellian distribution.

Assuming the eigenvalues are analytical at $\vec{k} = 0$, the smallest eigenvalue will be a continuation from $\lambda_0 = 0$ for small enough $\vec{k}$. Then the long term behaviour of the system will be dominated by this eigenvalue!

3.8.3 Spectrum of the linear operator

Introducing $\epsilon \ll 1$ for small wavevectors, we can write the linear operator, eq. (30), as

$$L_{\vec{k}} \Phi_{\vec{k}} = n_e n_i I[\Phi_{\vec{k}}] + i\epsilon \vec{k} \cdot \vec{c}\, \Phi_{\vec{k}}(\vec{c}, t) = L_0 + i\epsilon L_1 \ . \tag{33}$$

Considering the eigenvalue problem, eq. (3.36), for the smallest eigenvalue $\lambda \equiv \lambda_0(\vec{k})$ and expanding this eigenvalue and its eigenfunction Ψ in a perturbation series

$$\lambda = 0 + \epsilon\lambda_1 + \epsilon^2 \lambda_2 + \dots \tag{3.41}$$

$$\Psi = 1 + \epsilon \Psi_1 + \epsilon^2 \Psi_2 + \dots \ , \tag{3.42}$$

we can expand eq. (3.36)

$$(L_0 + i\epsilon L_1)\Psi = \lambda f_{\text{MB}}(\vec{c})\Psi \tag{3.36$'$}$$

$$(L_0 + i\epsilon L_1)(1 + \epsilon\Psi_1 + \epsilon^2\Psi_2 + \dots) = (\epsilon\lambda_1 + \epsilon^2\lambda_2 + \dots) f_{\text{MB}}(\vec{c})(1 + \epsilon\Psi_1 + \epsilon^2\Psi_2 + \dots) \ . \tag{34}$$

into powers of ϵ:

$$\epsilon^0 : \quad L_0 1 = 0 \tag{35}$$

$$\epsilon^1 : \quad L_0\Psi_1 + iL_1 1 = f_{\text{MB}}(\vec{c})\lambda_1 \tag{36}$$

$$\epsilon^2 : \quad L_0\Psi_2 + iL_1\Psi_1 = f_{\text{MB}}(\vec{c})[\lambda_1\Psi_1 + \lambda_2] \tag{37}$$

$$\epsilon^1 : \quad L_0\Psi_3 + iL_1\Psi_2 = f_{\text{MB}}(\vec{c})[\lambda_1\Psi_2 + \lambda_2\Psi_1 + \lambda_3] \ . \tag{38}$$

We solve now recursively and reorder the equations by moving the lower order functions with the operator L_1 to the other side. The first equation, eq. (35) is just trivially true: the integral operator of a constant vanishes. eq. (36) gives

$$L_0\Psi_1 = f_{\text{MB}}(\vec{c})\lambda_1 - iL_1 1 = f_{\text{MB}}(\vec{c})[\lambda_1 - i\vec{k}\cdot\vec{c}] \ . \tag{39}$$

This equation can only then have a solution, if

$$0 = (\Psi_0 = 1, \Psi_1) = \int d^3c\, f_{\text{MB}}(\vec{c})[\lambda_1 - i\vec{k}\cdot\vec{c}] = n_e\lambda_1 - 0 \ . \tag{40}$$

But that means $\lambda_1 = 0$, too! This simplifies eq. (39) to

$$L_0\Psi_1 = n_e n_i I[\Psi_1] = -i\vec{k}\cdot\vec{c}\, f_{\text{MB}}(\vec{c}) = -i\vec{k}\cdot\vec{c}\, n_e\, \hat{f}_{\text{MB}}(\vec{c}) \ , \tag{41}$$

which has apart from constant factors the same form as eq. (3.25)

$$I[\Phi] = \frac{q}{n_i k_B T} f_{\text{MB}}(\vec{c})\vec{c}\cdot\vec{E}_0 \ , \tag{3.25}$$

with the "solution", eq. (3.19), $\Phi(\vec{c}) = \phi(c)\vec{c}\cdot\vec{E}_0$. We can conclude,

$$\Psi_1 = -i\frac{k_B T}{n_i}(\vec{c}\cdot\vec{k})\hat{\phi} \ , \tag{3.46}$$

with $\hat{\phi}$ being the solution of eq. (3.26), will be a solution to our problem, eq. (41).

Now we can come to order ϵ^2, eq. (37):

$$L_0\Psi_2 = \lambda_2 f_{\text{MB}}(\vec{c}) - iL_1\Psi_1 \ . \tag{42}$$

Again, like with Ψ_1, this equation con only then have a solution, if

$$0 = (\Psi_0 = 1, \Psi_2) = \int d^3c\, f_{\text{MB}}(\vec{c})[\lambda_2 - i\vec{k}\cdot\vec{c}\,\Psi_1] = n_e\lambda_2 - i\int d^3c\, f_{\text{MB}}(\vec{c})\vec{k}\cdot\vec{c}\,\Psi_1 \ . \tag{43}$$

But that determines λ_2:

$$\begin{aligned}\lambda_2 &= \frac{i}{n_e}\int d^3c\, f_{\text{MB}}(\vec{c})L_1\Psi_1 = \frac{i}{n_e}\int d^3c\, f_{\text{MB}}(\vec{c})\frac{-ik_BT}{n_i}(\vec{c}\cdot\vec{k})^2\hat{\phi} = \frac{k_BT}{n_en_i}\frac{\vec{k}^2}{3}\int d^3c\, c^2 f_{\text{MB}}(\vec{c})\hat{\phi}\\ &= \frac{k_BT}{n_en_i}\frac{\vec{k}^2}{3}\frac{3n_i\sigma}{q^2} = \left(\frac{k_BT\sigma}{n_eq^2}\right)\vec{k}^2 \ .\end{aligned} \tag{3.48}$$

So our smallest eigenvalue is given by order ϵ^2:

$$\lambda_0(\vec{k}) = 0 + \epsilon\lambda_1 + \epsilon^2\lambda_2 + \cdots = \epsilon^2\left(\frac{k_BT\sigma}{n_eq^2}\right)\vec{k}^2 \ . \tag{44}$$

3.8.4 Diffusive behaviour

At large times all modes have relaxed to the smallest wavevector, which relaxes to the Maxwellian distribution ... in a diffusive manner: we get

$$\begin{aligned}\rho(\vec{r},t) &= \int d^3c\, f(\vec{r},\vec{c},t) = \int d^3c\, f_{\text{MB}}(\vec{c})[1 + \Phi_{\vec{k}}(\vec{c},t)e^{i\vec{k}\cdot\vec{r}}]\\ &= \int d^3c\, f_{\text{MB}}(\vec{c})[1 + (1 + \epsilon\Psi_1 + \epsilon^2\Psi_2 + \dots)e^{i\vec{k}\cdot\vec{r}}] \ .\end{aligned} \tag{45}$$

Integrating again eq. (3.1) over $d^3\vec{c}$ with $\vec{E} = 0$ we get

$$\begin{aligned}0 &= \int d^3c\left[\frac{\partial}{\partial t} + \vec{c}\cdot\frac{\partial}{\partial\vec{r}}\right]f[\vec{r},\vec{c};t]\\ &= \frac{\partial\rho}{\partial t} + \int d^3c\,[\vec{c}\cdot\nabla]\, f_{\text{MB}}(\vec{c})[1 + (1 + \epsilon\Psi_1 + \dots)e^{i\vec{k}\cdot\vec{r}}]\\ &= \frac{\partial\rho}{\partial t} + \nabla_j\int d^3c\, f_{\text{MB}}(\vec{c})[0 + (1 + \epsilon[-i\frac{k_BT}{n_i}(\vec{c}\cdot\vec{k})\hat{\phi}] + \dots)\vec{c}_je^{i\vec{k}\cdot\vec{r}}]\\ &= \frac{\partial\rho}{\partial t} + \nabla_j\int d^3c\, f_{\text{MB}}(\vec{c})(0 - i\epsilon[\frac{k_BT}{n_i}\vec{c}_k(-i\nabla_k)\hat{\phi}] + \dots)\vec{c}_je^{i\vec{k}\cdot\vec{r}}\\ &= \frac{\partial\rho}{\partial t} - \epsilon\int d^3c\, f_{\text{MB}}(\vec{c})\frac{k_BT}{n_i}(\vec{c}\cdot\vec{\nabla})^2\hat{\phi}e^{i\vec{k}\cdot\vec{r}}\end{aligned} \tag{46}$$

But the integral over d^3c in spherical coordinates, orienting the $\hat{z}$-axis along $\vec{r}$, reduces $(\vec{c}\cdot\vec{\nabla})$ to $c\cos\theta\nabla_{\vec{r}}$, resulting in

$$0 = \frac{\partial\rho}{\partial t} - \frac{k_BT}{n_i}\nabla^2\int d\varphi\, d(\cos\theta)c^2dc\, f_{\text{MB}}(\vec{c})c^2\hat{\phi}\epsilon e^{i\vec{k}\cdot\vec{r}} = \frac{\partial\rho}{\partial t} - \nabla^2\frac{k_BT}{n_i}\frac{4\pi}{3}\int c^2dc\,\rho\widehat{f}_{\text{MB}}(\vec{c})c^2\hat{\phi} \ , \tag{47}$$

where we included the exponential $e^{i\vec{k}\cdot\vec{r}}$ into the density ρ when going from f_{MB} to $\widehat{f}_{\text{MB}}$. But this is exactly the same integral that we could identify with σ, eq. (3.27), resulting in

$$\frac{\partial\rho}{\partial t} = \nabla^2\frac{k_BT}{n_i}\frac{1}{3}\rho\frac{3n_i\sigma}{q^2n_e} = \left(\frac{k_BT\sigma}{q^2n_e}\right)\nabla^2\rho = D\nabla^2\rho \tag{48}$$

3.8.5 Rigid hard spheres

Since the kinetic energy is conserved in the rigid hard sphere scattering, also the modulus $|\vec{c}|$ is conserved and a constant. Therefore any distribution $f(|\vec{c}|)$ is ok and will not relax to the Maxwellian distribution.

Analysing the linear operator $I[\Psi]$ in the subspace of functions, that only depend on the direction $\hat{c}$, reveals that $I[\Psi]$ is hermitian in this subspace. Therefore long term dynamics will show the same diffusive behaviour.

3.8.6 Time scales

For small values of the wavevectors, the eigenvalues will be continuous functions of $\vec{k}$ and will have a hierarchy:

$$0 = \lambda_0(\vec{k} = 0) < \lambda_0(\vec{k} \neq 0) \ll \lambda_1(\vec{k}) < \lambda_2(\vec{k}) < \dots \tag{49}$$

λ_0 corresponds to conserved "charges", $\lambda_{i>0}$ correspond to kinetic modes.

claim of the book: without spatial dependence, the timescale for λ_1 is the collision frequency ν. (see exercise 3.6.c)

- for $t \ll \nu^{-1}$ electrons move freely
  - ballistic regime
  - f is unchanged (as there are no collisions)
- $t_k \sim \nu^{-1}$ establishes the
  - kinetic regime
  - the kinetic modes relax to the **local** Maxwell distribution
- around $t \sim (Dk^2)^{-1} \gg \nu^{-1}$ the distribution is Maxwellian, but still inhomogeneous

  $\Rightarrow$ Hydrodynamic equations
- for $t \gg (Dk^2)^{-1}$ we have thermodynamic equilibrium:
  - Maxwell Boltzmann distribution with uniform density

condition for time scale separation: k is small enough, that $(Dk^2) \ll \nu$.

- taking $k = \frac{2\pi}{L}$ with the characteristic length scale L for density variations we can write $L \gg 2\pi\ell$, with ℓ being the mean free path.

$\Rightarrow$ time scale separation

$\Rightarrow$ Navier Stokes equations

  - the 5 collisional invariants of mass, momentum, and energy
  - give 5 null-eigenvalues that generate the hydrodynamic modes

3.9 The Chapman-Enskog method

uses time scale separation

1. step:

$$f(\vec{r}, \vec{c}, t) = h(\vec{c}; n(\vec{r}, t)) \tag{3.52}$$

2. introduce a small parameter ϵ
   - assuming that inhomogeneities are small

     $\Rightarrow$ gradient term is multiplied with ϵ
   - assuming the electric field is small

     $\Rightarrow$ $\vec{E}$ term is multiplied with ϵ
3. making a perturbation expansion for the distribution function

$$h = h_0 + \epsilon h_1 + \epsilon^2 h_2 + \dots \tag{3.53}$$

4. introducing scaled time variables:

$$t_0 = t \ , \qquad t_1 = \epsilon t \ , \qquad t_2 = \epsilon^2 t \ , \qquad \dots \tag{50}$$

and writing

$$n(\vec{r}, t) = n(\vec{r}, t_0, t_1, t_2, \dots) \tag{51}$$

then

$$\frac{\partial n}{\partial t} \to \frac{\partial n}{\partial t_0} + \epsilon \frac{\partial n}{\partial t_1} + \epsilon^2 \frac{\partial n}{\partial t_2} + \dots \tag{3.54}$$

and plugging all that into the Lorentz equation, eq. (3.1):

$$\left[\frac{\partial}{\partial t} + \epsilon \vec{c} \cdot \frac{\partial}{\partial \vec{r}} + \epsilon \frac{q}{m} \vec{E} \cdot \frac{\partial}{\partial \vec{c}}\right] h = \int [h'F' - hF] |\vec{c} - \vec{c}_1| \, b \, db \, d\psi \, d^3 c_1 \tag{3.1$'$}$$

- Integrating the r.h.s. over d^3c **always** yields zero due to charge conservation (i.e. the collisional invariant).
- expanding the l.h.s. in ϵ we get

$$\begin{aligned}
&\frac{\partial h_0}{\partial n}\left[\frac{\partial n}{\partial t_0} + \epsilon \frac{\partial n}{\partial t_1} + \epsilon^2 \frac{\partial n}{\partial t_2} + \dots\right] + \epsilon \frac{\partial h_1}{\partial n}\left[\frac{\partial n}{\partial t_0} + \epsilon \frac{\partial n}{\partial t_1} + \dots\right] + \epsilon^2 \frac{\partial h_2}{\partial n}\left[\frac{\partial n}{\partial t_0} + \dots\right] \\
&+ \epsilon \vec{c} \cdot \frac{\partial n}{\partial \vec{r}}\left[\frac{\partial h_0}{\partial n} + \epsilon \frac{\partial h_1}{\partial n} + \dots\right] + \epsilon \frac{q}{m} \vec{E} \cdot \left[\frac{\partial h_0}{\partial \vec{c}} + \epsilon \frac{\partial h_1}{\partial \vec{c}} + \dots\right] \\
=& \epsilon^0 \frac{\partial h_0}{\partial n} \frac{\partial n}{\partial t_0} && (52) \\
&+ \epsilon^1 \left[\frac{\partial h_0}{\partial n} \frac{\partial n}{\partial t_1} + \frac{\partial h_1}{\partial n} \frac{\partial n}{\partial t_0} + \vec{c} \cdot \frac{\partial n}{\partial \vec{r}} \frac{\partial h_0}{\partial n} + \frac{q}{m} \vec{E} \cdot \frac{\partial h_0}{\partial \vec{c}}\right] && (53) \\
&+ \epsilon^2 \left[\frac{\partial h_0}{\partial n} \frac{\partial n}{\partial t_2} + \frac{\partial h_1}{\partial n} \frac{\partial n}{\partial t_1} + \frac{\partial h_2}{\partial n} \frac{\partial n}{\partial t_0} + \vec{c} \cdot \frac{\partial n}{\partial \vec{r}} \frac{\partial h_1}{\partial n} + \frac{q}{m} \vec{E} \cdot \frac{\partial h_1}{\partial \vec{c}}\right] + \dots && (54)
\end{aligned}$$

  - order ϵ^0 integrated should be zero, telling us: $\frac{\partial n}{\partial t_0} = 0$
    $\Rightarrow$ we can make the ansatz:

$$h_0 = n(\vec{r}, t_1, t_2, \dots) \widehat{f}_{\mathrm{MB}}(c) \ . \tag{55}$$

  - when integrating the order ϵ^1 term, we see:
    * the second term is zero because $\frac{\partial n}{\partial t_0} = 0$
    * the third term is zero, because it is odd in $\vec{c}$
    * the fourth term is zero, as it is only a boundary term (and $\frac{\widehat{f}_{\mathrm{MB}}}{\partial \vec{c}}$ is odd in $\vec{c}$, too)

    $\Rightarrow$ $\frac{\partial n}{\partial t_1} = 0$, too!
    $\Rightarrow$ we can make the ansatz:

$$h_1 = h_0 \Phi(\vec{c}) = n(\vec{r}, t_2, \dots) \widehat{f}_{\mathrm{MB}}(c) \, \Phi(\vec{c}) \ . \tag{56}$$

- inserting this ansatz, eq. (56), into the r.h.s. of eq. (3.1$'$) we get

$$= \int n(\vec{r}, t_2, \dots) \widehat{f}_{\mathrm{MB}}(c) \, F[\Phi(\vec{c}\,') - \Phi(\vec{c})] |\vec{c} - \vec{c}_1| \, b \, db \, d\psi \, d^3 c_1 = -n \, I[\Phi] \ , \tag{57}$$

  since $\widehat{f}_{\mathrm{MB}}(c') \, F(c_1') = \widehat{f}_{\mathrm{MB}}(c) \, F(c_1)$.
- the full ϵ^1 equation with the ansatz eq. (56) becomes

$$\begin{aligned}
-n \, I[\Phi] &= \vec{c} \cdot \frac{\partial n}{\partial \vec{r}} \frac{\partial h_0}{\partial n} + \frac{q}{m} \vec{E} \cdot \frac{\partial h_0}{\partial \vec{c}} \ = \vec{c} \cdot (\vec{\nabla} n) \widehat{f}_{\mathrm{MB}}(c) + \frac{q}{m} \vec{E} \cdot \frac{\partial \widehat{f}_{\mathrm{MB}}(c)}{\partial \vec{c}} n && (58) \\
&= \widehat{f}_{\mathrm{MB}} \vec{c} \cdot (\vec{\nabla} n) + \frac{nq}{m} \vec{E} \cdot \frac{-m\vec{c}}{k_B T} \widehat{f}_{\mathrm{MB}} \ = -n \widehat{f}_{\mathrm{MB}} \frac{\vec{c}}{k_B T} \cdot \left[q\vec{E} - \frac{k_B T \vec{\nabla} n}{n}\right] \ =: -n \frac{\widehat{f}_{\mathrm{MB}}}{k_B T} \vec{c} \cdot \vec{G} \ ,
\end{aligned}$$

  which suggest due to linearity and isotropy the ansatz

$$\Phi(\vec{c}) = \hat{\phi} \, \vec{c} \cdot \vec{G} \ . \tag{59}$$

- inserting now eq. (56) with eq. (59) into the ϵ^2 equation we get

$$\frac{\partial h_0}{\partial n}\frac{\partial n}{\partial t_2} + \vec{c}\cdot\frac{\partial n}{\partial \vec{r}}\frac{\partial h_1}{\partial n} + \frac{q}{m}\vec{E}\cdot\frac{\partial h_1}{\partial \vec{c}} = -\,n\,I[\hat{\phi}\,\vec{c}\cdot\vec{G}] \ . \tag{60}$$

Multiplying with the charge q and integrating over d^3c gets rid of the third term, as it is again a surface term, and of the r.h.s. $I[\hat{\phi}\,\vec{c}\cdot\vec{G}]$, giving

$$0 = \frac{\partial}{\partial t_2}q\int d^3c\widehat{f}_{\rm MB}n + \vec{\nabla}q\int d^3c\,h_1\vec{c} = \frac{\partial \rho}{\partial t_2} + \vec{\nabla}\cdot\vec{J} \ , \tag{61}$$

but $\vec{J}$ can now be expressed as

$$\begin{aligned}\vec{J}_j =& q\int d^3c\,n\widehat{f}_{\rm MB}\hat{\phi}\,\vec{c}_j\vec{c}_k\left[q\vec{E}_k - \frac{k_BT\vec{\nabla}_k n}{n}\right]\\ =& q^2n\left[\int d^3c\,\widehat{f}_{\rm MB}\hat{\phi}\,\vec{c}_j\vec{c}_k\right]\vec{E}_k - qk_BT\left[\int d^3c\,\widehat{f}_{\rm MB}\hat{\phi}\,\vec{c}_j\vec{c}_k\right]\vec{\nabla}_k n \ .\end{aligned} \tag{62}$$

We can recognize the terms in the bracket as the conductivity tensor, eq. (3.20). Since we have an isotropic system, we can take the trace to obtain the scalar conductivity, eq. (3.21), or with the reduced functions, eq. (3.27):

$$\left[\int d^3c\,\widehat{f}_{\rm MB}\hat{\phi}\,\vec{c}_j\vec{c}_k\right] = \frac{\sigma}{q^2n} \ , \tag{63}$$

giving

$$\vec{J}_k = \sigma\vec{E}_k - D\vec{\nabla}_k\rho \ , \tag{64}$$

and with eq. (61)

$$0 = \frac{\partial\rho}{\partial t_2} + \vec{\nabla}_k(\sigma\vec{E}_k) - D\vec{\nabla}^2\rho \ , \tag{65}$$

which states the dynamics at the slow scale t_2.

3.10 Applications: bacterial suspensions, run-and-tumble motion

some bla bla about bacteria . . . E. coli follow a Lorentz-like distribution

$$\frac{\partial f}{\partial t} + V\hat{n}\cdot\vec{\nabla}f = \mu\int W(\hat{n}',\hat{n})f(\vec{r},\hat{n}',t)d^2\hat{n}' - \mu f(\vec{r},\hat{n},t) \ . \tag{3.63}$$

The first term on the r.h.s is the gain term, the second the loss term.

- W gives the probability of changing the direction of motion from $\hat{n}$ into $\hat{n}'$.

  ⇒ Therefore

$$\int W(\hat{n}',\hat{n})d^2\hat{n} = 1 \ . \tag{3.64}$$

- in an isotropic medium $W(\hat{n}',\hat{n}) = w(\hat{n}'\cdot\hat{n})$ with the flat distribution $W = \frac{1}{4\pi}$.
- eq. (3.63) is identical to the Lorentz equation for hard spheres
- with the identification $n_i\pi R^2|\vec{c}| = \mu$ we get diffusive motion for $t_D \gg \mu^{-1}$ with the diffusion coefficient $D = \frac{V^2}{3\mu}$.
- taking the more accurate $w(\hat{n}'\cdot\hat{n})$ (than the flat $W = 1/4\pi$) — expanding like in small wave vectors, i.e. sec. 3.8, or with the Chapman-Enskog method, i.e. sec 3.9 — we get

$$D = \frac{V^2}{3\mu(1-\alpha)} \ , \tag{3.66}$$

with

$$\alpha = \int(\hat{n}'\cdot\hat{n})w(\hat{n}'\cdot\hat{n})d^2\hat{n} = 2\pi\int_0^\pi \sin\theta\cos\theta w(\cos\theta)d\theta = 2\pi\int_{-1}^1 xw(x)dx \ . \tag{3.66}$$

$\alpha = 0$ gives isotropic tumbling. E. coli has $\alpha \sim 0.33$.

There is something like conductivity, called chemotaxis: the bacterial behaviour is modeled by making the tumbling probability μ dependent on the chemical gradient $\vec{\nabla}c$:

$$\mu \,\to\, \mu(\hat{n}\cdot\vec{\nabla}c) \;, \tag{66}$$

giving the kinetic model

$$\frac{\partial f}{\partial t} + V\hat{n}\cdot\vec{\nabla}f = \int \mu(\hat{n}'\cdot\vec{\nabla}c)w(\hat{n}'\cdot\hat{n})f(\vec{r},\hat{n}',t)\, d^2\hat{n}' - \mu(\hat{n}\cdot\vec{\nabla}c)f(\vec{r},\hat{n},t) \;. \tag{3.68}$$

When the gradient is small we can work in linear response theory and get the distribution

$$f(\vec{r},\hat{n},t) \;=\; f_0\left[1 + \frac{\mu_1\hat{n}\cdot\vec{\nabla}c}{\mu_0}\right] \;, \tag{3.69}$$

where μ is linearly expanded

$$\mu(\hat{n}\cdot\vec{\nabla}c) = \mu_0 - \mu_1\hat{n}\cdot\vec{\nabla}c \;, \tag{67}$$

giving the current

$$\vec{J} \;=\; \int V\hat{n}f(\vec{r},\hat{n},t)\, d^2\hat{n} \;=\; \frac{4\pi\mu_1 V}{3\mu_0}\vec{\nabla}c \;, \tag{3.70}$$

which is the chemotactic effect.