Problems

4.5. Hard sphere collision rule. The objective is to derive the collision rule for hard spheres of equal mass (4.7). Consider two spheres that at collision are joined by the vector $D\hat{n}$, where D is the contact distance. Because for hard spheres the collision is instantaneous, it can be modelled by a momentum transfer, $\Delta\vec{p}$. The spheres are smooth, implying that they do not exert tangential forces and the momentum transfer is parallel to $\hat{n}$. Imposing energy conservation, derive the collision rule.

$$\vec{c}\,' = \vec{c} + [(\vec{c} - \vec{c}_1)\cdot\hat{n}]\hat{n} \ , \quad \vec{c}_1' = \vec{c}_1 - [(\vec{c} - \vec{c}_1)\cdot\hat{n}]\hat{n} \tag{4.7}$$

1. In the collision of the two hard spheres we can split the momenta $\vec{p}$ in parts that are parallel to the direction of the vector connecting the two centers of the spheres ($\vec{p}_{\|}$) and parts that are orthogonal ($\vec{p}_{\perp}$): $\vec{p} = \vec{p}_{\|} + \vec{p}_{\perp}$. Then each sphere experiences the same scattering as if it would hit a plane orthogonal to the connecting vector $\hat{n}$, meaning that the component of the of the momentum parallel to $\hat{n}$ ($\vec{p}_{\|}$) will be reflected, giving the new (reflected) momentum $\vec{p}\,' = -\vec{p}_{\|} + \vec{p}_{\perp}$.

   - taking the vector $\hat{n}$ that describes the direction of the connection of the centers of the spheres as normalized, $\hat{n}\cdot\hat{n} = 1$, simplifies the abbreviations:

     $$\vec{p}_{\|} = (\vec{p}\cdot\hat{n})\hat{n} \qquad \text{and hence} \quad \vec{p}_{\perp} = \vec{p} - \vec{p}_{\|} = \vec{p} - (\vec{p}\cdot\hat{n})\hat{n} \ , \tag{1}$$

     giving the reflected momentum as

     $$\vec{p}\,' = -\vec{p}_{\|} + \vec{p}_{\perp} = -(\vec{p}\cdot\hat{n})\hat{n} + (\vec{p} - \vec{p}_{\|}) = \vec{p} - 2(\vec{p}\cdot\hat{n})\hat{n} \ . \tag{2}$$

   - Momentum conservation gives us:

     $$\vec{p} + \vec{p}_1 = \vec{p}\,' + \vec{p}_1' \ , \tag{3}$$

     telling us that we have to have for the other momentum:

     $$\vec{p}_1' = \vec{p} + \vec{p}_1 - \vec{p}\,' = \vec{p}_1 + 2(\vec{p}\cdot\hat{n})\hat{n} \ . \tag{4}$$

   - The analysis of the parallel and perpendicular parts also tells us, that the total momentum will be composed of the unchanging parts, identifying $\hat{n}$ to be orthogonal to $\vec{P}$.
   - then it follows, that $\vec{p}_{\perp}$ should be parallel to $\vec{P}$, giving us

     $$\vec{p}_{\perp} = \frac{(\vec{p}\cdot\vec{P})}{\vec{P}^2}\vec{P} \ . \tag{5}$$

2. going to Jacobi coordinates, i.e. total momentum (in the non-relativistic regime)

   $$(m_0 + m_1)\vec{C} = M\vec{C} = \vec{P} = \vec{p}_0 + \vec{p}_1 = m_0\vec{c} + m_1\vec{c}_1 = m_0\vec{c}\,' + m_1\vec{c}_1' \ , \tag{6}$$

   and relative velocities,

   $$\vec{g} = \vec{c} - \vec{c}_1 \qquad \vec{g}\,' = \vec{c}\,' - \vec{c}_1' \ , \tag{7}$$

   we can express the velocities by $\vec{C}$, $\vec{g}$, and $\vec{g}\,'$:

   $$M\vec{C} + m_1\vec{g} = m_0\vec{c} + m_1\vec{c}_1 + m_1\vec{c} - m_1\vec{c}_1 = M\vec{c} \quad \Rightarrow \quad \vec{c} = \vec{C} + \tfrac{m_1}{M}\vec{g} \tag{8}$$

   $$M\vec{C} - m_0\vec{g} = m_0\vec{c} + m_1\vec{c}_1 - (m_0\vec{c} - m_0\vec{c}_1) = M\vec{c}_1 \quad \Rightarrow \quad \vec{c}_1 = \vec{C} - \tfrac{m_0}{M}\vec{g} \ , \tag{9}$$

   and analogously for the primed ones:

   $$\vec{c}\,' = \vec{C} + \tfrac{m_1}{M}\vec{g}\,' \qquad \vec{c}_1' = \vec{C} - \tfrac{m_0}{M}\vec{g}\,' \ . \tag{10}$$

   - going to the center of momentum frame (CM frame) with velocities $\vec{v}$, defined by

     $$0 = m_0\vec{v} + m_1\vec{v}_1 = m_0\vec{v}\,' + m_1\vec{v}_1' \ , \tag{11}$$

     we get $\vec{v} = \vec{c} - \vec{C}$, or explicitely

     $$\vec{v} = \tfrac{m_1}{M}\vec{g} \qquad \vec{v}_1 = -\tfrac{m_0}{M}\vec{g} \qquad \vec{v}\,' = \tfrac{m_1}{M}\vec{g}\,' \qquad \vec{v}_1' = -\tfrac{m_0}{M}\vec{g}\,' \ . \tag{12}$$

- looking at energy conservation in the CM frame

$$\frac{1}{2}m_0\vec{v}^2 + \frac{1}{2}m_1\vec{v}_1^2 = \frac{1}{2}m_0(\frac{m_1}{M}\vec{g})^2 + \frac{1}{2}m_1(-\frac{m_0}{M}\vec{g})^2 = \frac{1}{2}\frac{m_0m_1^2+m_0^2m_1}{M^2}\vec{g}^2 = \frac{1}{2}\frac{m_0m_1}{M}\vec{g}^2 = \frac{1}{2}\mu\vec{g}^2$$
$$= \frac{1}{2}m_0\vec{c}'^2 + \frac{1}{2}m_1\vec{c}_1'^2 = \cdots = \frac{1}{2}\mu\vec{g}'^2 \ , \quad (13)$$

giving immediately $|\vec{g}| = |\vec{g}'|$.

3. applying the splitting from 1. to the relative velocity $\vec{g}$ we get

$$\vec{g} = \vec{g}_{||} + \vec{g}_\perp \quad \text{with} \quad \vec{g}_{||} = (\vec{g}\cdot\hat{n})\hat{n} \quad \text{and} \quad \vec{g}_\perp = \vec{g} - (\vec{g}\cdot\hat{n})\hat{n} \ . \quad (14)$$

Using the assumption $\vec{p}_0' = \vec{p}_0 + \Delta\vec{p}$ (as written in the text) we have

$$\vec{c}' = \vec{c} + \frac{1}{m_0}\Delta\vec{p} \ , \quad \text{and} \quad \vec{c}_1' = \vec{c}_1 - \frac{1}{m_1}\Delta\vec{p} \ . \quad (15)$$

Subtracting these equations we get the equation for the relative velocity:

$$\vec{g}' = \vec{c}' - \vec{c}_1' = \vec{c} + \frac{1}{m_0}\Delta\vec{p} - (\vec{c}_1 - \frac{1}{m_1}\Delta\vec{p}) = \vec{g} + (\frac{1}{m_0} + \frac{1}{m_1})\Delta\vec{p} = \vec{g} + \frac{1}{\mu}\Delta\vec{p} =: \vec{g} + \vec{t} \ . \quad (16)$$

Combining this now with the splitting (14) we have

$$\vec{g}' = \vec{g}_{||}' + \vec{g}_\perp' \quad (17)$$

with

$$\vec{g}_{||}' = (\vec{g}'\cdot\hat{n})\hat{n} = [(\vec{g}\cdot\hat{n}) + (\vec{t}\cdot\hat{n})]\hat{n} \quad (18)$$

and

$$\vec{g}_\perp' = \vec{g}' - (\vec{g}'\cdot\hat{n})\hat{n} = \vec{g} + \vec{t} - [(\vec{g}\cdot\hat{n}) + (\vec{t}\cdot\hat{n})]\hat{n} \ . \quad (19)$$

But we also have the conservation of $|\vec{g}|$, eq. (13):

$$|\vec{g}|^2 = |\vec{g}_{||}|^2 + 2\vec{g}_{||}\cdot\vec{g}_\perp + |\vec{g}_\perp|^2 = |\vec{g}_{||}|^2 + |\vec{g}_\perp|^2 \ , \quad (20)$$

since $\vec{g}_{||}$ and $\vec{g}_\perp$ are by construction orthogonal. Inserting the splitting vector $\hat{n}$ just shows consistency:

$$|\vec{g}_{||}|^2 + |\vec{g}_\perp|^2 = (\vec{g}\cdot\hat{n})^2|\hat{n}|^2 + |\vec{g} - (\vec{g}\cdot\hat{n})\hat{n}|^2 = (\vec{g}\cdot\hat{n})^2 + |\vec{g}|^2 - 2(\vec{g}\cdot\hat{n})^2 + (\vec{g}\cdot\hat{n})^2|\hat{n}|^2 = |\vec{g}|^2 \ , \quad (21)$$

leading to

$$|\vec{g}|^2 = |\vec{g}'|^2 = |\vec{g}_{||}'|^2 + |\vec{g}_\perp'|^2 = |\vec{g} + \vec{t}|^2 = |\vec{g}|^2 + 2(\vec{g}\cdot\vec{t})^2 + |\vec{t}|^2 \ , \quad (22)$$

which tells us, that the projection of the transfer on $\vec{g}$ has to be negative:

$$(\vec{t}\cdot\vec{g}) = -\frac{1}{2}\vec{t}^2 < 0 \ . \quad (23)$$

4. using the analysis of 1. again on $\vec{g}$ tells us, that $|\vec{g}_\perp|$ should not change:

$$|\vec{g}_\perp|^2 = |\vec{g}|^2 - (\vec{g}\cdot\hat{n})^2 = |\vec{g}_\perp'|^2 = |\vec{g} + \vec{t} - [(\vec{g}\cdot\hat{n}) + (\vec{t}\cdot\hat{n})]\hat{n}|^2 = |\vec{g} + \vec{t}|^2 - [(\vec{g}\cdot\hat{n}) + (\vec{t}\cdot\hat{n})]^2$$
$$= |\vec{g}|^2 - (\vec{g}\cdot\hat{n})^2 - 2(\vec{g}\cdot\hat{n})(\vec{t}\cdot\hat{n}) - (\vec{t}\cdot\hat{n})^2 = |\vec{g}_\perp|^2 - (\vec{t}\cdot\hat{n})[(2\vec{g} + \vec{t})\cdot\hat{n}]$$
$$= |\vec{g}_\perp|^2 - [(\vec{g}' - \vec{g})\cdot\hat{n}][(\vec{g}' + \vec{g})\cdot\hat{n}] \ , \quad (24)$$

telling us that

$$\vec{g}_{||}' = \pm\vec{g}_{||} \ . \quad (25)$$

Using again the argument, that $\vec{g}_\perp$ should be conserved, we have to pick the minus sign in (25) and get the requested answer:

$$\vec{g}' = \vec{g}_{||}' + \vec{g}_\perp' = -\vec{g}_{||} + \vec{g}_\perp = -(\vec{g}\cdot\hat{n})\hat{n} + \vec{g} - (\vec{g}\cdot\hat{n})\hat{n} = \vec{g} - 2(\vec{g}\cdot\hat{n})\hat{n} \ , \quad (26)$$

and with equal masses from (8), (9), and (10):

$$\vec{c}' = \vec{C} + \frac{1}{2}\vec{g}' = \frac{1}{2}(\vec{c} + \vec{c}_1) + \frac{1}{2}[\vec{g} - 2(\vec{g}\cdot\hat{n})\hat{n}] = \vec{c} - [(\vec{c} - \vec{c}_1)\cdot\hat{n}]\hat{n} \quad (27)$$
$$\vec{c}_1' = \vec{C} - \frac{1}{2}\vec{g}' = \frac{1}{2}(\vec{c} + \vec{c}_1) - \frac{1}{2}[\vec{g} - 2(\vec{g}\cdot\hat{n})\hat{n}] = \vec{c}_1 + [(\vec{c} - \vec{c}_1)\cdot\hat{n}]\hat{n} \ . \quad (28)$$