Problems

4.11. BGK model. Show that the BGK model has the same collisional invariants as the Boltzmann equation and that an H-theorem exists.

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \vec{c}\cdot\frac{\partial f}{\partial \vec{r}} + \frac{\vec{F}}{m}\cdot\frac{\partial f}{\partial \vec{c}} = J_{\rm BGK}[f] := \nu\{f_{\rm MB}(\vec{c};n[f],\vec{v}[f],T[f]) - f(\vec{r},\vec{c},t)\}\ , \tag{4.41}$$

with

$$n[f] = \int f(\vec{r},\vec{c},t)\,d^3\vec{c}\ , \tag{4.42}$$

$$\vec{v}[f] = \frac{1}{n(\vec{r},t)}\int \vec{c}\,f(\vec{r},\vec{c})\,d^3\vec{c}\ , \tag{4.43}$$

$$\frac{3}{2}k_{\rm B}T[f] = \frac{1}{n(\vec{r},t)}\int \frac{m}{2}(\vec{c}-\vec{v})^2\,f(\vec{r},\vec{c})\,d^3\vec{c}\ . \tag{4.44}$$

- A collisional invariant $\varphi(\vec{c})$ is defined as the function that gives zero when multiplied with eq.(4.41) and integrated over $d^3\vec{c}$:

$$\begin{aligned} 0 = \int \varphi(\vec{c})\,J_{\rm BGK}[f]\,d^3\vec{c} &:= \nu \int \varphi(\vec{c})\,\{f_{\rm MB}(\vec{c};n[f],\vec{v}[f],T[f]) - f(\vec{r},\vec{c},t)\}\,d^3\vec{c} \\ &= \nu\int \varphi(\vec{c})\,\{n[f]\left(\tfrac{m}{2\pi k_{\rm B}T}\right)^{3/2} e^{-m(\vec{c}-\vec{v})^2/(2k_{\rm B}T)} - f(\vec{r},\vec{c},t)\}\,d^3\vec{c}\ . \end{aligned} \tag{1}$$

- the functions to be tested are $\varphi = \{m, m\vec{c}, \frac{1}{2}m(\vec{c}-\vec{v})^2\}$:

1:

$$\begin{aligned} &\int \{n[f]\left(\tfrac{m}{2\pi k_{\rm B}T}\right)^{3/2} e^{-m(\vec{c}-\vec{v})^2/(2k_{\rm B}T)} - f(\vec{r},\vec{c},t)\}\,d^3\vec{c} \\ &= n[f]\left(\int \left(\tfrac{m}{2\pi k_{\rm B}T}\right)^{3/2} e^{-m(\vec{c}')^2/(2k_{\rm B}T)}\,d^3\vec{c}'\right) - n[f] \\ &= n[f]\left(\int \pi^{-3/2}e^{-\vec{x}^2}\,d^3\vec{x}\right) - n[f] \;=\; n[f]\left(\pi^{-3/2}4\pi\int_0^\infty e^{-r^2}\,r^2\,dr\right) - n[f] = 0\ . \end{aligned} \tag{2}$$

$\vec{c}$:

$$\begin{aligned} &\int \vec{c}\{n[f]\left(\tfrac{m}{2\pi k_{\rm B}T}\right)^{3/2} e^{-m(\vec{c}-\vec{v})^2/(2k_{\rm B}T)} - f(\vec{r},\vec{c},t)\}\,d^3\vec{c} \\ &= n[f]\left(\int (\vec{c}' + \vec{v}[f])\left(\tfrac{m}{2\pi k_{\rm B}T}\right)^{3/2} e^{-m(\vec{c}')^2/(2k_{\rm B}T)}\,d^3\vec{c}'\right) - n[f]\vec{v}[f] \\ &= n[f]\left(0 + \vec{v}[f]\int \pi^{-3/2}e^{-\vec{x}^2}\,d^3\vec{x}\right) - n[f]\vec{v}[f] \\ &= n[f]\vec{v}[f]\left(\pi^{-3/2}4\pi\int_0^\infty e^{-r^2}\,r^2\,dr\right) - n[f]\vec{v}[f] = 0\ . \end{aligned} \tag{3}$$

$m(\vec{c}-\vec{v})^2$:

$$\begin{aligned} &\int m(\vec{c}-\vec{v})^2\{n[f]\left(\tfrac{m}{2\pi k_{\rm B}T}\right)^{3/2} e^{-m(\vec{c}-\vec{v})^2/(2k_{\rm B}T)} - f(\vec{r},\vec{c},t)\}\,d^3\vec{c} \\ &= n[f]\left(\int m\vec{c}'^2\left(\tfrac{m}{2\pi k_{\rm B}T}\right)^{3/2} e^{-m(\vec{c}')^2/(2k_{\rm B}T)}\,d^3\vec{c}'\right) - n[f]3k_{\rm B}T[f] \\ &= n[f]m\left(\int \pi^{-3/2}\frac{2k_{\rm B}T[f]}{m}\vec{x}^2 e^{-\vec{x}^2}\,d^3\vec{x}\right) - 3k_{\rm B}n[f]T[f] \\ &= n[f]\pi^{-3/2}4\pi\left(2k_{\rm B}T[f]\int_0^\infty e^{-r^2}\,r^4\,dr\right) - 3k_{\rm B}n[f]T[f] \\ &= n[f]4\pi^{-1/2}\left(2k_{\rm B}T[f]\frac{3\sqrt{\pi}}{8}\right) - 3k_{\rm B}n[f]T[f] = 0\ . \end{aligned} \tag{4}$$

For the H-theorem we have to find a function H, of which the total time derivative is always negativ.

- Using the same ansatz as for the Boltzman equation

$$H[f] = \int f(\vec{r}, \vec{c}, t) \ln[f(\vec{r}, \vec{c}, t)/f_0] \, d^3\vec{c} \ , \tag{4.23}$$

 we get analogously

$$\frac{dH}{dt} = \int [\ln[f(\vec{r}, \vec{c}, t)/f_0] + 1] \frac{d f(\vec{r}, \vec{c}, t)}{dt} d^3\vec{c} = \int [\ln[f(\vec{r}, \vec{c}, t)/f_0] + 1] \, J_{\mathrm{BGK}}[f] \, d^3\vec{c} \ , \tag{4.24}$$

 with a steady distribution, that does not depend on time. We know that $f_{\mathrm{MB}}(\vec{c}; n[f], \vec{v}[f], T[f])$ does not depend on time. If we set now for the definition of our H-function $f_0 = f_{\mathrm{MB}}(\vec{c}; n[f], \vec{v}[f], T[f])$ we get

$$\begin{aligned} \frac{dH}{dt} &= \nu \int [\ln f(\vec{r}, \vec{c}, t) - \ln f_0 + 1] \, \{f_{\mathrm{MB}}(\vec{c}; n[f], \vec{v}[f], T[f]) - f(\vec{r}, \vec{c}, t)\} \, d^3\vec{c} \\ &= -\nu \int [\ln f_{\mathrm{MB}}(\vec{c}; n[f], \vec{v}[f], T[f]) - \ln f(\vec{r}, \vec{c}, t)] \, [f_{\mathrm{MB}}(\vec{c}; n[f], \vec{v}[f], T[f]) - f(\vec{r}, \vec{c}, t)] \, d^3\vec{c} \\ &= -\nu \int [\ln x - \ln y] \, [x - y] \, d^3\vec{c} < 0 \ , \end{aligned} \tag{5}$$

 since we had already, that 1 is a collisional invariant, eq. (2) and its integral vanishes.

- The rest of the argument for the H-theorem is exactly the same as for the Boltzmann equation:
 1. the integrand for H, $f \ln[f/f_0]$, is bounded from below.
 2. so H can only then become unbounded from below, if the integral for large $|\vec{c}|$ diverges.
 3. that requires that $f \ln[f/f_0] c^2 < -Ac^{-1}$ for some positive constant A.
 4. the total energy is finite and conserved, limiting $f < Bc^{-5}$ for some positive constant B.
 5. conditions 3. and 4. together imply $f < f_0 e^{-Ac^2/B}$.
 6. this result 5. gives a finite H, which contradicts the assumption, that H can be unbounded from below.
 7. therefore H will obtain its minimum value, given by the collisional invariants, i.e. by f_{MB}.