Problems

1.5. Establish (a) Jacobi's identity (1.26e), and (b) the relation (1.26g).

$$[A,[B,F]]+[B,[F,A]]+[F,[A,B]]=0 \tag{1.26e}$$

$$[A,B(A)]=0 \tag{1.26g}$$

a) algebraically trivial:

$$\begin{aligned}
&[A,[B,F]]+[B,[F,A]]+[F,[A,B]]\\
=&ABF-AFB-BFA+FBA\\
&+BFA-FBA-FAB+BAF\\
&+FAB-BAF-ABF+AFB\\
=&0
\end{aligned} \tag{1}$$

When writing it for the Poisson bracket

$$[B,F]=\sum_\ell \frac{\partial B}{\partial q_\ell}\frac{\partial F}{\partial p_\ell}-\frac{\partial F}{\partial q_\ell}\frac{\partial B}{\partial p_\ell} \quad , \tag{2}$$

we get many more terms because of the derivatives:

$$\begin{aligned}
&[A,[B,F]]\\
=&\sum_k \frac{\partial A}{\partial q_k}\frac{\partial}{\partial p_k}[B,F]-\frac{\partial}{\partial q_k}[B,F]\frac{\partial A}{\partial p_k}\\
=&\sum_{k,\ell} \frac{\partial A}{\partial q_k}\frac{\partial}{\partial p_k}\left(\frac{\partial B}{\partial q_\ell}\frac{\partial F}{\partial p_\ell}-\frac{\partial F}{\partial q_\ell}\frac{\partial B}{\partial p_\ell}\right)-\frac{\partial}{\partial q_k}\left(\frac{\partial B}{\partial q_\ell}\frac{\partial F}{\partial p_\ell}-\frac{\partial F}{\partial q_\ell}\frac{\partial B}{\partial p_\ell}\right)\frac{\partial A}{\partial p_k}\\
=&\sum_{k,\ell} \frac{\partial A}{\partial q_k}\left(\frac{\partial^2 B}{\partial p_k\partial q_\ell}\frac{\partial F}{\partial p_\ell}+\frac{\partial B}{\partial q_\ell}\frac{\partial^2 F}{\partial p_k\partial p_\ell}-\frac{\partial^2 F}{\partial p_k\partial q_\ell}\frac{\partial B}{\partial p_\ell}-\frac{\partial F}{\partial q_\ell}\frac{\partial^2 B}{\partial p_k\partial p_\ell}\right)\\
&-\frac{\partial A}{\partial p_k}\left(\frac{\partial^2 B}{\partial q_k\partial q_\ell}\frac{\partial F}{\partial p_\ell}+\frac{\partial B}{\partial q_\ell}\frac{\partial^2 F}{\partial q_k\partial p_\ell}-\frac{\partial^2 F}{\partial q_k\partial q_\ell}\frac{\partial B}{\partial p_\ell}-\frac{\partial F}{\partial q_\ell}\frac{\partial^2 B}{\partial q_k\partial p_\ell}\right) \quad .
\end{aligned} \tag{3}$$

When summing over the cyclic permutations we just have to remember, that the summation indices can be renamed and that the derivative is symmetric, then all terms cancel again, like in eq. (1): the renaming cancels all terms with the derivatives $\frac{\partial^2}{\partial q_i\partial p_j}$ and the symmetry $\frac{\partial^2}{\partial x_i\partial x_j}=\frac{\partial^2}{\partial x_j\partial x_i}$ does the rest. Mathematica file provided ...

b) treating the function as a Taylor series

$$B(A)=\sum_n b_n A^n \quad , \tag{4}$$

we can write eq. (1.26g) as

$$[A,B(A)]=\sum_n b_n[A,A^n] \quad . \tag{5}$$

The Leibnitz rule, eq. (1.26d), gives

$$[A,A^n]=[A,A\cdot A^{n-1}]=[A,A]A^{n-1}+A[A,A^{n-1}]=\cdots=n[A,A]A^{n-1}=0 \quad , \tag{6}$$

as does the explicit Poisson bracket:

$$[A,A^n]=\sum_\ell \frac{\partial A}{\partial q_\ell}\frac{\partial A^n}{\partial p_\ell}-\frac{\partial A^n}{\partial q_\ell}\frac{\partial A}{\partial p_\ell}=\sum_\ell \frac{\partial A}{\partial q_\ell}nA^{n-1}\frac{\partial A}{\partial p_\ell}-nA^{n-1}\frac{\partial A}{\partial q_\ell}\frac{\partial A}{\partial p_\ell}=0 \quad . \tag{7}$$

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In[6]:= PB[a_, b_] :=
          Module[{i}, partial[a, q[i]] partial[b, p[i]] - partial[b, q[i]] partial[a, p[i]]]

In[7]:= PB[A, B]

Out[7]= partial[A, q[i$790]] partial[B, p[i$790]] - partial[A, p[i$790]] partial[B, q[i$790]]

In[14]:= jacobi = (Expand[PB[A, PB[B, F]] + PB[B, PB[F, A]] + PB[F, PB[A, B]]]) /.
            partial[partial[a_, b_], c_] ↦ partial2[a, Sort[{b, c}]]

Out[14]= partial[B, q[i$6416]] partial[F, q[i$6415]] partial2[A, {p[i$6415], p[i$6416]}] -
   partial[B, p[i$6416]] partial[F, q[i$6415]] partial2[A, {p[i$6415], q[i$6416]}] -
   partial[B, q[i$6416]] partial[F, p[i$6415]] partial2[A, {p[i$6416], q[i$6415]}] -
   partial[B, q[i$6417]] partial[F, q[i$6418]] partial2[A, {p[i$6417], p[i$6418]}] +
   partial[B, q[i$6417]] partial[F, p[i$6418]] partial2[A, {p[i$6417], q[i$6418]}] +
   partial[B, p[i$6417]] partial[F, q[i$6418]] partial2[A, {p[i$6418], q[i$6417]}] +
   partial[B, p[i$6416]] partial[F, p[i$6415]] partial2[A, {q[i$6415], q[i$6416]}] -
   partial[B, p[i$6417]] partial[F, p[i$6418]] partial2[A, {q[i$6417], q[i$6418]}] -
   partial[A, q[i$6414]] partial[F, q[i$6413]] partial2[B, {p[i$6413], p[i$6414]}] +
   partial[A, p[i$6414]] partial[F, q[i$6413]] partial2[B, {p[i$6413], q[i$6414]}] +
   partial[A, q[i$6414]] partial[F, p[i$6413]] partial2[B, {p[i$6414], q[i$6413]}] +
   partial[A, q[i$6417]] partial[F, q[i$6418]] partial2[B, {p[i$6417], p[i$6418]}] -
   partial[A, q[i$6417]] partial[F, p[i$6418]] partial2[B, {p[i$6417], q[i$6418]}] -
   partial[A, p[i$6417]] partial[F, q[i$6418]] partial2[B, {p[i$6418], q[i$6417]}] -
   partial[A, p[i$6414]] partial[F, p[i$6413]] partial2[B, {q[i$6413], q[i$6414]}] +
   partial[A, p[i$6417]] partial[F, p[i$6418]] partial2[B, {q[i$6417], q[i$6418]}] +
   partial[A, q[i$6414]] partial[B, q[i$6413]] partial2[F, {p[i$6413], p[i$6414]}] -
   partial[A, p[i$6414]] partial[B, q[i$6413]] partial2[F, {p[i$6413], q[i$6414]}] -
   partial[A, q[i$6414]] partial[B, p[i$6413]] partial2[F, {p[i$6414], q[i$6413]}] -
   partial[A, q[i$6415]] partial[B, q[i$6416]] partial2[F, {p[i$6415], p[i$6416]}] +
   partial[A, q[i$6415]] partial[B, p[i$6416]] partial2[F, {p[i$6415], q[i$6416]}] +
   partial[A, p[i$6415]] partial[B, q[i$6416]] partial2[F, {p[i$6416], q[i$6415]}] +
   partial[A, p[i$6414]] partial[B, p[i$6413]] partial2[F, {q[i$6413], q[i$6414]}] -
   partial[A, p[i$6415]] partial[B, p[i$6416]] partial2[F, {q[i$6415], q[i$6416]}]

In[19]:= jacobi1 = SumDeltaInternalAll [
           jacobi /. {partial[A, i_[i1_]] :> partial[A, i[a]] * MyIndexDelta[a, i1]} /.
              {partial[B, i_[i1_]] :> partial[B, i[b]] * MyIndexDelta[b, i1]} /.
            {partial[F, i_[i1_]] :> partial[F, i[c]] * MyIndexDelta[c, i1]}]

Out[19]= -partial[B, q[b]] partial[F, q[c]] partial2[A, {p[b], p[c]}] +
   partial[B, q[b]] partial[F, q[c]] partial2[A, {p[c], p[b]}] -
   partial[B, p[b]] partial[F, p[c]] partial2[A, {q[b], q[c]}] +
   partial[B, p[b]] partial[F, p[c]] partial2[A, {q[c], q[b]}] +
   partial[A, q[a]] partial[F, q[c]] partial2[B, {p[a], p[c]}] -
   partial[A, q[a]] partial[F, q[c]] partial2[B, {p[c], p[a]}] +
   partial[A, p[a]] partial[F, p[c]] partial2[B, {q[a], q[c]}] -
   partial[A, p[a]] partial[F, p[c]] partial2[B, {q[c], q[a]}] -
   partial[A, q[a]] partial[B, q[b]] partial2[F, {p[a], p[b]}] +
   partial[A, q[a]] partial[B, q[b]] partial2[F, {p[b], p[a]}] -
   partial[A, p[a]] partial[B, p[b]] partial2[F, {q[a], q[b]}] +
   partial[A, p[a]] partial[B, p[b]] partial2[F, {q[b], q[a]}]
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In[23]:= Collect[jacobi1, {partial[A, _] partial[B, _],
              partial[A, _] partial[F, _], partial[B, _] partial[F, _]}]

Out[23]= partial[B, q[b]] partial[F, q[c]]
           (-partial2[A, {p[b], p[c]}] + partial2[A, {p[c], p[b]}]) + partial[B, p[b]]
           partial[F, p[c]] (-partial2[A, {q[b], q[c]}] + partial2[A, {q[c], q[b]}]) +
         partial[A, q[a]] partial[F, q[c]]
           (partial2[B, {p[a], p[c]}] - partial2[B, {p[c], p[a]}]) + partial[A, p[a]]
           partial[F, p[c]] (partial2[B, {q[a], q[c]}] - partial2[B, {q[c], q[a]}]) +
         partial[A, q[a]] partial[B, q[b]]
           (-partial2[F, {p[a], p[b]}] + partial2[F, {p[b], p[a]}]) + partial[A, p[a]]
           partial[B, p[b]] (-partial2[F, {q[a], q[b]}] + partial2[F, {q[b], q[a]}])

In[20]:= jacobi1/. {partial2[a_, {b_, c_}] :> partial2[a, Sort[{b, c}]]}

Out[20]= 0
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