Problems

1.48. The initial value of an ensemble density is given on the spherical surface in Γ-space, $\sum p_i^2 + \sum q_i^2 = a^2$ to be

$$D_0(q,p) = \sum q_i^2 \exp\left[-\sum p_i^2\right] = e^{-\sum p_i^2} \sum q_i^2$$

System points of the ensemble are comprised of N independent harmonic oscillators with potential $V = q^2/2$ and unit mass. What is the solution $D(q,p,t)$ to the Liouville equation?

- According to the "Solution from trajectories", p.26, eq.(4.22)

$$D(q,p,t) = D_0[q - \tilde{q}(t), p - \tilde{p}(t)] \ , \tag{4.22}$$

where $\tilde{q}(t)$ and $\tilde{p}(t)$ are the "homogeneous" solution.

⇒ we have only to find the solutions to the harmonic oscillator with unit mass and potential $V = q^2/2$,

$$H = \tfrac{1}{2}p^2 + \tfrac{1}{2}q^2 \tag{1}$$

which was done in exercise 1.13 with $m = k = \omega = 1$:

$$q = \sin(t+\varphi) \quad \text{and} \quad p = \cos(t+\varphi) \ , \tag{2}$$

giving the "homogeneous" solutions

$$\begin{aligned} \tilde{q}(t) =&\ q - q_0 = q - q(0) = \sin(t+\varphi) - \sin\varphi \\ \text{and} \quad \tilde{p}(t) =&\ \cos(t+\varphi) - \cos\varphi \ . \end{aligned} \tag{3}$$

The different harmonic oscillators and only differ by their initial value:

$$q_i = \sin(t+\varphi_i) \quad \text{and} \quad p_i = \cos(t+\varphi_i) \ , \tag{4}$$

giving a constant value of the Hamiltonian:

$$H_i = \tfrac{1}{2}p_i^2 + \tfrac{1}{2}q_i^2 = \tfrac{1}{2} \ . \tag{5}$$

The initial surface then tells us

$$a^2 = \sum p_i^2 + \sum q_i^2 = \sum (p_i^2 + q_i^2) = \sum 1 \ , \tag{6}$$

that we have a^2 harmonic oscillators with no restrictions on φ_i and

$$\begin{aligned} D(q_i,p_i,t) &= D_0[q_i - \tilde{q}_i(t), p_i - \tilde{p}_i(t)] \ = \ e^{-\sum p_i^2} \sum q_i^2 \\ &= e^{-\sum [p_i - \cos(t+\varphi_i) + \cos\varphi_i]^2} \sum [q_i - \sin(t+\varphi_i) + \sin\varphi_i]^2 \ . \end{aligned} \tag{7}$$

... and I still fail to understand the goal of this task ...