Problems

1.24. A collection of four identical particles moving in one dimension are known to be in the following state at a given time $t > 0$. One particle is moving with velocity v_1^0 and another with v_2^0. Both these particles were at the origin $x = 0$ at $t = 0$. The remaining two particles are stationary at x_3^0 and x_4^0, respectively.

(a) Write down a determinantal joint-probability distribution that describes this state.

(b) Obtain an expression for $f_1(x, v)$ from your answer to (a).

a) We can write the individual distributions for the particles as

$$\varphi_1(x, v) = \delta(v - v_1^0)\delta(x - v_1^0 t) \tag{1}$$
$$\varphi_2(x, v) = \delta(v - v_2^0)\delta(x - v_2^0 t) \tag{2}$$
$$\varphi_3(x, v) = \delta(v)\delta(x - x_3^0) \tag{3}$$
$$\varphi_4(x, v) = \delta(v)\delta(x - x_4^0) \ . \tag{4}$$

Then $f_4 = f_4(x_1, v_1, x_2, v_2, x_3, v_3, x_4, v_4)$ can be written as

$$\begin{aligned} f_4 =& \frac{1}{4!} \sum_{\mathcal{P}(abcd)} \varphi_1(x_a, v_a)\varphi_2(x_b, v_b)\varphi_3(x_c, v_c)\varphi_4(x_d, v_d) \\ =& \frac{1}{4!} \sum_{\mathcal{P}(abcd)} \varphi_a(x_1, v_1)\varphi_b(x_2, v_2)\varphi_c(x_3, v_3)\varphi_d(x_4, v_4) \ , \end{aligned} \tag{5}$$

where $\mathcal{P}(abcd)$ indicates all permutations of $(abcd) = (1234)$.

The book writes this as the "determinant with only plus signs".

b) Since

$$f_1(x_1, v_1) := \int d2\, d3\, d4\, f_4 \ , \tag{6}$$

and the integral

$$\int dk\, \varphi_i(x_k, v_k) = \int dx_k\, dp_k \varphi_i(x_k, v_k) = 1 \ , \tag{7}$$

we get

$$\begin{aligned} f_1(x, v) =& \int d2\, d3\, d4\, \frac{1}{4!} \sum_{\mathcal{P}(abcd)} \varphi_a(x, v)\varphi_b(x_2, v_2)\varphi_c(x_3, v_3)\varphi_d(x_4, v_4) \\ =& \frac{1}{4!} \sum_{a,\mathcal{P}(bcd)} \varphi_a(x, v) \int d2\, d3\, d4\, \varphi_b(x_2, v_2)\varphi_c(x_3, v_3)\varphi_d(x_4, v_4) \\ =& \frac{1}{4!} \sum_a \varphi_a(x, v) \sum_{\mathcal{P}(bcd)} 1 \\ =& \frac{1}{4!} \sum_a \varphi_a(x, v) \ \ 3! \\ =& \frac{1}{4} \left[\varphi(x, v) + \varphi_2(x, v) + \varphi_3(x, v) + \varphi_4(x, v)\right] \ . \end{aligned} \tag{8}$$