## Problems

**1.20.** Show that the Poisson brackets of two Cartesian components of the angular momentum $L$ of a particle about a specified origin obey the relation

$$[L_i, L_j] = L_k \tag{1}$$

where $(i, j, k)$ are the variables $(x, y, z)$ in cyclic order.

- We can write the angular momentum of a particle as

$$\vec{L} := \vec{r} \times \vec{p} \quad \text{or in index notation: } L_j := \epsilon_{jk\ell} r_k p_\ell := \epsilon_{jk\ell} q_k p_\ell \ , \tag{2}$$

  when writing the radius as the generalised coordinate. Then we have (using Einstein's summation convention; i.e. ommiting the $\sum$ symbols for sums over indices that appear exactly two times)

$$\begin{aligned}
&[L_i, L_j] \\
&= \sum_h \frac{\partial L_i}{\partial q_h}\frac{\partial L_j}{\partial p_h} - \frac{\partial L_j}{\partial q_h}\frac{\partial L_i}{\partial p_h} \\
&= \epsilon_{imn}\epsilon_{jk\ell}\left(\frac{\partial(q_m p_n)}{\partial q_h}\frac{\partial(q_k p_\ell)}{\partial p_h} - \frac{\partial(q_k p_\ell)}{\partial q_h}\frac{\partial(q_m p_n)}{\partial p_h}\right) \\
&= \epsilon_{imn}\epsilon_{jk\ell}\left(\delta_{hm} p_n \delta_{h\ell} q_k - \delta_{hk} p_\ell \delta_{hn} q_m\right) \\
&= \epsilon_{imn}\epsilon_{jk\ell}\left(\delta_{m\ell} p_n q_k - \delta_{nk} p_\ell q_m\right) \\
&= p_n q_k \epsilon_{i\ell n}\epsilon_{jk\ell} - p_\ell q_m \epsilon_{imk}\epsilon_{jk\ell} \\
&= p_b q_a \epsilon_{i\ell b}\epsilon_{ja\ell} - p_b q_a \epsilon_{iak}\epsilon_{jkb} \\
&= p_b q_a (-\epsilon_{ib\ell}\epsilon_{ja\ell} + \epsilon_{iak}\epsilon_{jbk}) \\
&= p_b q_a \left(-(\delta_{ij}\delta_{ab} - \delta_{ia}\delta_{jb}) + (\delta_{ij}\delta_{ab} - \delta_{ib}\delta_{ja})\right) \\
&= q_a p_b \left(\delta_{ia}\delta_{jb} - \delta_{ib}\delta_{ja}\right) \\
&= q_a p_b \epsilon_{kij}\epsilon_{kab} \\
&= \epsilon_{ijk}\left(\epsilon_{kab} q_a p_b\right) \\
&= \epsilon_{ijk} L_k \ .
\end{aligned} \tag{3}$$