Exercise 1.41

Typo
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A system with N degrees of freedom has coordinates $\{x_i\}$ and momenta $\{p_i\}$, $i = 1, \dots, N$. Consider that $\{x_i\}$ is written as a column vector $\vec{x}$. A transformation to new coordinates $\vec{x}'$ is given by

$$\vec{x}' = \bar{\bar{M}} \vec{x}$$

where the elements of the $N \times N$ matrix, $\bar{\bar{M}}$, are constants.

a) If we write

$$\vec{p}' = \bar{\bar{Q}} \vec{p}$$

what is the relation between $\bar{\bar{Q}}$ and $\bar{\bar{M}}$ that ensures that the transformation is canonical?

b) Under what conditions will the transformation be canonical if $\bar{\bar{Q}} = \bar{\bar{M}}$?

A transformation is canonical if

$$[q'_i, q'_j] = [p'_i, p'_j] = 0$$
$$[q'_i, p'_j] = \delta_{ij}$$

For a) we have

$$q'_i = \sum_{m=1}^{N} M_{im} q_m \quad , \quad p'_i = \sum_{m=1}^{N} Q_{im} p_m$$

So, the conditions become

$$[q'_i, q'_j] = \sum_{k=1}^{N} \left(\frac{\partial q'_i}{\partial q_k} \underbrace{\frac{\partial q'_j}{\partial p_k}}_{0} - \frac{\partial q'_j}{\partial q_k} \underbrace{\frac{\partial q'_i}{\partial p_k}}_{0} \right) = 0$$

$$[p'_i, p'_j] = \sum_{k=1}^{N} \left(\underbrace{\frac{\partial p'_i}{\partial q_k}}_{0} \frac{\partial p'_j}{\partial p_k} - \underbrace{\frac{\partial p'_j}{\partial q_k}}_{0} \frac{\partial p'_i}{\partial p_k} \right) = 0$$

$$[q'_i, p'_j] = \sum_{k=1}^{N} \left(\frac{\partial q'_i}{\partial q_k} \frac{\partial p'_j}{\partial p_k} - \underbrace{\frac{\partial p'_j}{\partial q_k}}_{0} \underbrace{\frac{\partial q'_i}{\partial p_k}}_{0} \right) =$$

$$= \sum_{k=1}^{N} M_{ik} Q_{jk} = \delta_{ij}$$

In matrix notation we have

$$\cancel{\bar{\bar{M}} \cdot \bar{\bar{Q}}^T = \bar{\bar{I}}} \qquad \bar{\bar{M}} \cdot \bar{\bar{Q}}^T = \bar{\bar{I}} \implies \bar{\bar{Q}} = (\bar{\bar{M}}^T)^{-1} = (\bar{\bar{M}}^{-1})^T$$

So, for b) we would have $\bar{\bar{M}}^T \cdot \bar{\bar{M}} = \bar{\bar{I}} \implies \bar{\bar{M}}^T = \bar{\bar{M}}^{-1}$.
That is, $\bar{\bar{M}}$ is an orthogonal matrix.