Exercise 1.27

Show that the difference equation

$$P(l,n) = \frac{1}{2}\left[P(l+1, n-1) + P(l-1, n-1)\right] \qquad (1)$$

Types Ex. 1.27

(here l is displacement and $n = \frac{t}{\Delta t}$ counts time)

gives the diffusion equation

$$\frac{\partial}{\partial t} P(x,t) = D \frac{\partial^2}{\partial x^2} P(x,t)$$

where time t and displacement x are related to the integers n and l through the interval constants δ_t and δ_x as

$$n = \frac{t}{\delta_t} \quad , \quad l = \frac{x}{\delta_x}$$

and

$$D = \lim_{\delta_t, \delta_x \to 0} \frac{\delta_x^2}{\delta_t}$$

is assumed to be a finite parameter.

First we introduce $P(l,n) = P\left(\frac{x}{\delta_x}, \frac{t}{\delta_t}\right) = \bar{P}(x,t)$.
Now we drop the bar for convenience. Then, rewriting (1)

$$P(x,t) = \frac{1}{2}\left[P(x+\delta_x, t-\delta_t) + P(x-\delta_x, t-\delta_t)\right] \qquad (2)$$

Now we expand $P(x \pm \delta_x, t-\delta_t)$ in Taylor series about x

$$P(x \pm \delta_x, t-\delta_t) = P(x \cancel{\pm\delta_x}, t-\delta_t) \pm \delta_x \frac{\partial P(x, t-\delta_t)}{\partial x} + \frac{1}{2}\delta_x^2 \frac{\partial^2 P(x, t-\delta_t)}{\partial x^2}$$

Inserting the expansions in (2) we get

$$P(x,t) = P(x, t-\delta_t) + \frac{1}{2}\delta_x^2 \frac{\partial^2 P(x, t-\delta_t)}{\partial x^2}$$

Now we subtract $P(x, t-\delta_t)$ and divide by δ_t. We get

$$\frac{P(x,t) - P(x,t-\delta_t)}{\delta_t} = \frac{\delta_x^2}{\delta_t} \frac{\partial^2 P(x, t-\delta_t)}{\partial x^2}$$

Taking the limits $\delta_x, \delta_t \to 0$ we obtain

$$\frac{\partial}{\partial t} P(x,t) = D \frac{\partial^2}{\partial x^2} P(x,t)$$

This is the Kramers - Moyal expansion.