Exercise 1.10

Show that the Poisson bracket relation $[q,p]=1$ is invariant under ~~p~~ the transformation generated by

$$G_2(q,p') = q\sin p'$$

$$p = \frac{\partial G_2}{\partial q} \quad ; \quad q' = \frac{\partial G_2}{\partial p'}$$

$$p = \sin p' \Rightarrow p' = \arcsin p$$

$$q' = q\cos p' = q\cos(\arcsin p) = q\sqrt{1-p^2} \Rightarrow q = \frac{q'}{\cos p'}$$

$$[q,p]_{q',p'} = \frac{\partial q}{\partial q'}\frac{\partial p}{\partial p'} - \frac{\partial p}{\partial q'}\frac{\partial q}{\partial p'} = \frac{1}{\cos p'}\cdot\cos p' - 0 = 1$$

Exercise 1.14

a) Show that the equality

$$p_x\,dx + p_y\,dy + p_z\,dz = p_r\,dr + p_\theta\,d\theta + p_\varphi\,d\varphi \qquad (*)$$

follows from the transformation equations from Cartesian to spherical coordinates.

b) Argue that this transformation is canonical.

We have

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases} \qquad \begin{aligned} \dot{x} &= \dot{r}\sin\theta\cos\varphi + r\dot{\theta}\cos\theta\cos\varphi - r\dot{\varphi}\sin\theta\sin\varphi \\ \dot{y} &= \dot{r}\sin\theta\sin\varphi + r\dot{\theta}\cos\theta\sin\varphi + r\dot{\varphi}\sin\theta\cos\varphi \\ \dot{z} &= \dot{r}\cos\theta - r\dot{\theta}\sin\theta \end{aligned}$$

We can also write $(*)$ as

$$\underbrace{\frac{\partial L}{\partial \dot{x}}dx + \frac{\partial L}{\partial \dot{y}}dy + \frac{\partial L}{\partial \dot{z}}dz}_{I} = \underbrace{\frac{\partial L}{\partial \dot{r}}dr + \frac{\partial L}{\partial \dot{\theta}}d\theta + \frac{\partial L}{\partial \dot{\varphi}}d\varphi}_{II}$$

$$L = \frac{m}{2}\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right) \quad ; \quad L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2\right)$$

$$II = m\dot{r}\,dr + mr^2\dot{\theta}\,d\theta + mr^2\sin^2\theta\,d\varphi$$

$dx = \sin\theta\cos\varphi\,dr$

not sure, but this looks OK

$$I = m\dot{x}\,dx + m\dot{y}\,dy + m\dot{z}\,dz = mdt\{\dot{x}^2+\dot{y}^2+\dot{z}^2\}$$
$$= mdt\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\dot{\varphi}^2\right) = II$$

Hmm...

$$\cancel{T = \frac{m}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2) \; ; \; dT = m\dot{x}\,d\dot{x} + m\dot{y}\,d\dot{y} + m\dot{z}\,d\dot{z} \; ;}$$

$\cancel{\frac{d}{dt}\left(p_x \dot{x} + p_y \dot{y} + p_z \dot{z}\right)}$

Regarding the canonical character ...

We have

$$\sum_i p_i \, dq_i = \sum_i p_i' \, dq_i'$$

If we assume

$$G_2^{(p_i', q)} = \sum_i p_i' \, q_i'(q) \qquad q_i' = \frac{\partial G_2}{\partial p_i'}, \quad p_i = \frac{\partial G_2}{\partial q_i}$$

$$\frac{\partial}{\partial q_i} \sum_j p_j' \, q_j'(q) = \sum_j p_j' \frac{\partial q_j'}{\partial q_i}$$

then we should have

$$\sum_i p_i \, dq_i + \sum_i q_i' \, dp_i' = d\, G_2(q, p')$$

$$\sum_i p_i \, dq_i = \sum_i \frac{\partial G_2}{\partial q_i} dq_i \; ; \quad \sum_i p_i' \, q_i'(q) = \cancel{G}$$

$$\sum_i p_i \, dq_i' = d\Big(\sum_i p_i' q_i'\Big) - \sum_i q_i' \, dp_i'$$

$$\sum_i p_i \, dq_i - \sum_i p_i' \, dq_i' = \sum_i p_i' \frac{\partial G_2}{\partial q_i} dq_i - \sum_i p_i' \, dq_i'$$

$$= \sum_i \sum_j p_j' \frac{\partial q_j'}{\partial q_i} dq_i - \sum_i p_i' \, dq_i' = \sum_j p_j' \, dq_j' - \sum_i p_i' \, dq_i' = 0$$

So, $(*)$ is consistent with canonical transformation.

Exercise 1.26 Show that the probability distribution

$$P(\ell, n) = \frac{n!}{[\frac{1}{2}(n+\ell)]! \, [\frac{1}{2}(n-\ell)]!} p^{(n+\ell)/2} q^{(n-\ell)/2}$$

satisfies

Typo 1.23a

$$P(\ell, n) = q \, P(\ell+1, n-1) + p \, P(\ell-1, n-1)$$

$$q \, P(\ell+1, n-1) + p \, P(\ell-1, n-1) =$$

$$= q \frac{(n-1)!}{[\frac{1}{2}(n+\ell)]! \, [\frac{1}{2}(n-\ell-2)]!} p^{(n+\ell)/2} q^{(n-\ell-2)/2} +$$

$$+ p \frac{(n-1)!}{[\frac{1}{2}(n+\ell-2)]! \, [\frac{1}{2}(n-\ell)]!} p^{(n+\ell-2)/2} q^{(n-\ell)/2} =$$

$$= (n-1)! \cdot p^{(n+\ell)/2} q^{(n-\ell)/2} \left\{ \frac{1}{[\frac{1}{2}(n+\ell)]! \, [\frac{1}{2}(n-\ell-2)]!} + \frac{1}{[\frac{1}{2}(n+\ell-2)]! \, [\frac{1}{2}(n-\ell)]!} \right\} \circledast$$

$$\{ \; \} = \left\{ \frac{\frac{n-\ell}{2}}{[\frac{1}{2}(n+\ell)]! \, [\frac{1}{2}(n-\ell)]!} + \frac{\frac{n+\ell}{2}}{[\frac{1}{2}(n+\ell)]! \, [\frac{1}{2}(n-\ell)]} \right\} = \frac{n}{[\frac{1}{2}(n+\ell)]! \, [\frac{1}{2}(n-\ell)]!}$$

$$\circledast \; n! \, p^{(n+\ell)/2} q^{(n-\ell)/2} \frac{1}{[\frac{1}{2}(n+\ell)]! \, [\frac{1}{2}(n-\ell)]!}$$

So, proven.

Exercise 1.28

a) What is the equation of motion in Γ-space for the ensemble density function D that corresponds to the fact that system points are neither created nor destroyed.

b) How does this equation differ from the Liouville equation?

The equation is

$$\frac{\partial}{\partial t} D + \nabla \cdot (\vec{u} D) = 0$$

which is equivalent to the Liouville equation.